Completeness of FOL

We did have a look at a Hilbert-style axiom system to define a deductive consequence relation for Classical Propositional Logic so as to get the Soundness and Completeness results. Now, we would like to lift this idea to get the same for FOL.

Syntax of FOL $(\mathcal{C}, \mathcal{F}, \mathcal{P})$:

Terms : $t \in \mathcal{T} := c \in \mathcal{C} \mid x \in \mathcal{V} \mid f_i^k t_1 t_2 \cdots t_k$

Formulas : $\varphi, \psi \in F := p_i^m t_1 t_2 \cdots t_m \mid t = t' \mid \neg\varphi \mid \varphi \to \psi \mid \forall x_i \varphi$.

We are looking for $\Gamma \vdash \varphi$ s.t. :

Soundness : if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Completeness : if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

- The idea is to propose some axioms and rules to define $\vdash$.
Then to show soundness, it is enough to show that all (substitutional instances of) axioms are valid and that all (substitutional instances of) inference rules preserve semantic consequence.
Then to show soundness we apply induction on the length of the proof.

- For completeness, we proceed with the contrapositive form, that is, we assume $\Gamma \nvdash \varphi$ and show $\Gamma \not\models \varphi$

- We introduced the notion of consistency: A set of formulas Γ is said to be consistent if there is no formula φ s.t. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

Otherwise, Γ is said to be inconsistent.

- $\Gamma \nvdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is consistent. (H.W.)

- Also, $\Gamma \nvDash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.

- Thus to show completeness it is enough to show that: Every consistent set of formulas is satisfiable.

Now, what we saw from the proof structure of propositional logic?

Step 1: Extend a consistent set Γ to a consistent and complete set Δ.

Step 2: Define a model (valuation) based on Δ: $V_\Delta(p) = 1$ iff $p \in \Delta$.

Step 3: Prove the truth lemma: for all formulas α, $V_\Delta(\alpha) = 1$ iff $\alpha \in \Delta$.

Then we have that Δ is satisfiable and so, Γ is satisfiable.

Again, the proof structure of compactness theorem of FOL suggests us the following:

Step 1: Given a consistent set of formulas Γ, we extend it to a consistent, complete and witness-fulfilled set of formulas Δ.

Step 2: Define a model $\mathcal{M}_\Delta$ based on Δ:

- Define $t \sim t'$ iff $t = t' \in \Delta$. Show that this is a congruence relation
- $([t_1], [t_2], \ldots, [t_n]) \in \mathcal{I}(P)$ iff $Pt_1t_2\cdots t_n \in \Delta$.

Step 3: Prove the truth lemma: for all formulas φ, $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Then, we have that Δ is satisfiable and hence Γ is satisfiable.

Let us first make an observation regarding 'adding witnesses'.

We say the Γ is witness-fulfilled if for all formulas of the form $\exists x \varphi$, if $\exists x \varphi \in \Gamma$ then we also have $\varphi[t/x] \in \Gamma$ for some term t.

Now, how do we find such term?

We used the Gödel trick: expand language L to L' with new constant symbols and add them as witnesses

Lemma: Let Γ be a set of formulas and φ be a formula s.t. c does not occur in Γ or φ. Then if $\Gamma \models \varphi[c/x]$ then $\Gamma \models \forall x \varphi$.

Finally at the end, we constructed an L'-model for our fin-sat Γ and then restricted it to an L-structure to get an L-model.

An observation: There is no need to expand the language. New variable work just as well as constant symbols.

Variables: Let $V = \{x_1, x_2, \ldots\}$.

What do we do?

- In the given consistent set Γ that we start with, replace every variable x_i by x_{2i}. Now, all the odd-indexed variables are unused.
- We now show that the new set of formulas thus formed, say Γ', is consistent as well.
- In the Lindenbaum construction, whenever we come across a formula of the form $\exists x \varphi$, we add $\varphi[y/x]$ where y is the least unused variable
- We need to prove that $\varphi[y/x]$ can be added consistently: this is given by an analog of the Lemma mentioned above stated in terms of consistency.

What we discussed above suggests the following main steps need to prove our result: (Every consistent set is satisfiable)

- Define $t \sim t'$ iff $t = t' \in \Delta$. Show that this is a congruence relation.

- For any φ, x, t there exists φ' s.t. $\vdash \varphi \Leftrightarrow \varphi'$ and t is substitutable for x in φ'.

- Let Γ be a consistent set of formulas and φ a formula s.t. the variable y does not occur in Γ or φ. Then, if $\Gamma \vdash \varphi[y/x]$ then $\Gamma \vdash \forall x \varphi$.

We need to add axioms and rules to prove all these.

Next query: What are these axioms and rules?

What does the propositional system give us?

Axioms :

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
3. $(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

Rule :

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$ Modus Ponens (MP)

We note that the axioms are validities and the rule preserves consequence in FOL as well.

We also note that these axioms are only schema, they stand for infinitely many formulas obtained by substitutional instances. But these instances are first order formulas

Example : $\forall x Px \to (Qyz \to \forall x Px)$ is an instance of Ax 1.

- We see that first order formulas can come with free variables as well.

- So, we let the schema represent universalisations as well.

$\downarrow$

we can always prefix any substitutional instance by a (possibly empty) string of universal quantifiers.

<u>example</u>: The universalisations of a formula Pxy are Pxy, $\forall x Pxy$, $\forall y Pxy$, $\forall x \forall y Pxy$.

Now we are ready to define our deductive consequence relation in FOL.

Let Γ be a set formulas and φ be a formula. We say $\Gamma \vdash \varphi$ if there is a finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$, s.t. $\varphi_n = \varphi$ and each φ_i is

either a member of Γ

or a universalisation of a substitutional instance of an axiom,

or there is an inference rule of

The form $\dfrac{\varphi_j \,,\ \varphi_j \to \varphi_i}{\varphi_i}$ (M.P.)

with φ_i as conclusion. (that is, if the premise of an instantiation of M.P. are earlier in the proof sequence, then the conclusion can be used in the proof)

Let us now move on to proving the required steps.

Step 1 : We define $t \sim t'$ iff $t \doteq t' \in \Delta$
We have to show that $\sim$ is a congruence.

We have that Δ is consistent and complete. We can show:
- if $\vdash \varphi$ then $\varphi \in \Delta$ (H.W.)

Axioms needed :

- $t \doteq t$
- $(t \doteq t') \to (t' \doteq t)$
- $(t_1 \doteq t_2) \to ((t_2 \doteq t_3) \to (t_1 \doteq t_3))$

- $(t_1 = t_1') \to ((t_2 = t_2') \to \cdots ((t_n = t_n') \to f^n t_1 t_2 \cdots t_n = f^n t_1' t_2' \cdots t_n') \cdots)$
- $(t_1 = t_1') \to ((t_2 = t_2') \to \cdots ((t_n = t_n') \to (P^n t_1 t_2 \cdots t_n \Leftrightarrow P^n t_1' t_2' \cdots t_n')) \cdots)$.

As it turns out, we can have a more general version!

- $t \equiv t$
- $(t \equiv t') \to (\varphi \Leftrightarrow \varphi')$, where φ' is obtained by replacing some occurences of t by t' and some occurences of t' by t.

H.W. Use these axioms to prove the congruence of $\sim$.

The main ingredients that we would look into are:

(i) alphabetic equivalents result

(ii) generalisation of constants result.

Let us now deal with the result on alphabetic equivalents. We want to show: For any φ, x, t, there exists φ' s.t. $\vdash \varphi \leftrightarrow \varphi'$ and t is substitutable of x in φ'.

- How does one construct such a φ'?

By renaming bound variables.

- Rename all the variables in t having bound occurrences in φ.

- In addition, we need to show that the equivalence of the resulting formula can be derived in the axiom system.

- It suffices to show the following: $\vdash \forall x \varphi \leftrightarrow \forall y \varphi[y/x]$, where y is a new variable not occurring in $\forall x \varphi$.

Let us try to prove the result now.

We first prove $\vdash \forall x \varphi \to \forall y \varphi[y/x]$.

- Since y is new, it is of course substitutable for x in φ. Thus, $\models \forall x \varphi \to \varphi[y/x]$. Hence, this can be added an axiom. So, we have

$$\vdash \forall x \varphi \to \varphi[y/x]$$

By universalisation we have,

$$\vdash \forall y (\forall x \varphi \to \varphi[y/x])$$

Now, if we could distribute $\forall$ over $\to$, we would have,

$$\forall y \forall x \varphi \to \forall y \varphi[y/x]$$

But we have $\models \forall x (\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$

So, we can have an axiom:

$$\vdash \forall x (\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$$

So, we have,

$$\vdash \forall y (\forall x \varphi \to \varphi[y/x]) \to (\forall y \forall x \varphi \to \forall y \varphi[y/x])$$

Then we get by M.P. :

$\vdash \forall y \forall x \varphi \to \forall y \varphi[y/x]$

So we need, $\vdash \forall x \varphi \to \forall y \forall x \varphi$

Now, we have, $\vdash \forall x \varphi \to \forall y \forall x \varphi$

since y does not occur in $\forall x \varphi$.

Thus the corresponding axiom will do the job.

Conversely, $\vdash \forall y \varphi[y/x] \to (\varphi[y/x])[x/y]$

as no occurrence of x is free in $\varphi[y/x]$.

Then, $\vdash \forall y \varphi[y/x] \to \varphi$, and so,

$\vdash \forall x(\forall y \varphi[y/x] \to \varphi)$. So, we have

$\vdash \forall y \varphi[y/x] \to \forall x \varphi$. (Why?).

This completes the proof.

Now, what are the axioms that we have needed to get this result?

- $\forall x \varphi \to \varphi[t/x]$, for any term t substitutable of x in φ
- $\varphi \to \forall x \varphi$, where $x \notin FV(\varphi)$
- $\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$

The nice part is that these are all that we need !!

A formal proof of the result above

To prove $\vdash \forall x\, \varphi \to \forall y\, \varphi[y/x]$, where y is a new variable not occurring in $\forall x\, \varphi$.

1. $\vdash \forall x\, \varphi \to \varphi[y/x]$, by first quantifier axiom.

2. $\vdash \forall y\, (\forall x\, \varphi \to \varphi[y/x])$, universalisation

3. $\vdash \forall y \forall x\, \varphi \to \forall y\, \varphi[y/x]$, from 2., using third quantifier axiom and M.P.

4. $\vdash \forall x\, \varphi \to \forall y \forall x\, \varphi$, since $y \notin FV(\forall x\, \varphi)$, by second quantifier axiom

5. $\vdash \forall x\, \varphi \to \forall y\, \varphi[y/x]$, from 4,3, using propositional reasoning.

In general, we can always rename variables by new ones, so that derivability is preserved.

- Suppose $\Gamma \vdash \varphi$ and y is a variable not occurring in $\Gamma \cup \{\varphi\}$. Then, we have, $\Gamma[y/x] \vdash \varphi[y/x]$. This can be proved by applying induction on the length of the proof. (H.W.)

This also shows that Γ is consistent iff $\Gamma[y/x]$ is consistent.

Now we move to another generalisation result: Let Γ be a consistent set of formulas and φ be a formula such that the variable y does not occur in Γ or φ. Then if $\Gamma \vdash \varphi[y/x]$ then $\Gamma \vdash \forall x \varphi$.

Proof. We prove this by applying induction on the length of the proof of $\varphi[y/x]$ from Γ. Let $\psi_1, \psi_2, \ldots, \psi_n$ be a proof of $\varphi[y/x]$.

Claim : $\Gamma \vdash \forall y\, \psi_i$ for all i.

Assuming this claim, we get $\Gamma \vdash \forall y\, \varphi[y/x]$ and hence, by alphabetic renaming, we will have $\Gamma \vdash \forall x\, \varphi$. So, we will complete the proof by proving the claim.

We prove the claim by applying induction on i.

To prove : $\Gamma \vdash \forall y\, \psi_i$, $1 \leq i \leq m$.

Base Case : the proof length is 1.

Then, ψ_i is either an axiom or, $\psi_i \in \Gamma$, which does not have any occurence of the variable y. In the former case, we have $\vdash \psi_i$ and so, $\vdash \forall y\, \psi_i$ by universalisation. In the latter case, we have $\vdash \psi_i \to \forall y\, \psi_i$ since, ψ_i, being in Γ has no occurence of y. Then since, $\Gamma \vdash \psi_i$, $\Gamma \vdash \forall y\, \psi_i$.

Induction Hypothesis : Suppose the result holds good for proof lengths $\leq m$.

Induction Step. It is possible that φ_i is derived using M.P. So we have, φ_j and $\varphi_k := \varphi_j \to \varphi_i$ with both $j, k < m$

By induction hypothesis, $\Gamma \vdash \forall y (\varphi_j \to \varphi_i)$ and $\Gamma \vdash \forall y \varphi_j$. Apply distributivity axiom and MP, and we are done.

Let us now move on to the proof of completeness theorem. We have to show that: if Γ is a consistent set of formulas, then it is satisfiable.

Step 1: Extend Γ to a set Δ which is consistent, complete and witness-fulfilled.

Step 2: Define a model M_Δ based on Δ:
- Define $t \sim t'$ iff $t = t' \in \Delta$. Show that $\sim$ is a congruence
- Add $([t_1], \dots, [t_m])$ to the interpretation of P^m iff $P^m t_1 t_2 \dots t_m \in \Delta$.

Step 3 : Prove the truth lemma :

for all φ, $\mathcal{M}_\Delta \models \varphi$ iff $\varphi \in \Delta$.

Then Δ is satisfiable and hence Γ is satisfiable.

The first step :

Let $V = \{x_1, x_2, \ldots\}$

- In Γ, replace every variable x_i by x_{2i}. Then all the odd indexed variables are unused. We show that the new set Γ' is consistent.

Suppose not. Let Γ_1 be a finite subset that is inconsistent. Now, let x_{2n} be the first variable not occuring in Γ_1. Obtain Γ_2 by replacing in every formula in Γ_1, each occurence of x_{2j} by x_{2n+j}.

Now, by the proposition on renaming

of variables, Γ_2 is also inconsistent.

Then, obtain a set Γ'' by replacing in every formula of Γ_2 each occurrence of x_{2n+j} by x_j. Then, Γ'' becomes an inconsistent subset of Γ, which was itself a consistent set, a contradiction.

So, we have Γ' is consistent.

This completes the proof.

- The Lindenbaum construction:

Extend the consistent set Γ' to a consistent, complete and witness fulfilled set Δ. Let V_{odd} be the countable set of variables not used in Γ'. Enumerate the formulas of the language: $\varphi_0, \varphi_1, \varphi_2 \ldots$

Define a sequence of sets: $\Delta_0 = \Gamma'$,

$\Delta_1, \Delta_2, \ldots\ldots$ as follows :

Let $\Delta_k' = \Delta_k \cup \{\varphi_k\}$.

- Suppose that φ_k is of the form $\exists x \psi$. Define :

$\Delta_{k+1} = \Delta_k' \cup \{\psi[y_j/x]\}$ if Δ_k' is consistent, y_j is the least variable in V_{odd}, not occuring in Δ_k'.

$= \Delta_k$, otherwise.

- if φ_k is not of the form $\exists x \psi$, define, $\Delta_{k+1} = \Delta_k'$ if Δ_k' is consistent

$= \Delta_k$, otherwise.

$\Delta = \bigcup_k \Delta_k$. We can show that Δ is consistent, complete and witness fullfilled. (H.W. Complete this part!)

The second step :

We now do the model construction

Define M_Δ based on Δ.

- First define $t \sim t'$ iff $t \doteq t' \in \Delta$.
  By equality axioms, this $\sim$ relation is a congruence.

- Define $D_\Delta = T/\sim$, the quotient of $\sim$

- The interpretation of constant symbols : $c \to [c]$

- The interpretation of function symbols : $I(f^k)$ takes $[t_1], \dots [t_k]$ to $[f^k t_1 t_2 \dots t_k]$

- The assignment $g : x \to [x]$

- We show, $g(t) = [t]$, for all terms t.

- The interpretation of the predicate symbols P^m is given by :

$([t_1], \ldots [t_m]) \in I(P^m)$ iff $P^m t_1 \ldots t_m \in \Delta$.

This completes the definition of the model.

The third step:

We need to prove the truth lemma.
To show: For all φ, $M_\Delta \models \varphi$ iff $\varphi \in \Delta$.

We prove this by induction on the size of φ.

Base case: $\varphi := t = t'$. This case is given by the definition of the model: $t \equiv t' \in \Delta$ iff $[t] = [t']$.
The case $\varphi := P t_1 \ldots t_m$ is also given by the construction of the model. We assume the usual I.H.

Induction Step: For the propositional (boolean) connectives, the results follow from the fact that Δ is a consistent and complete set.

The only case that remains to be shown is the quantifier case. We need to show:

$M_\Delta \models \forall x \varphi$ iff $\forall x \varphi \in \Delta$.

Proof: Suppose $\forall x \varphi \notin \Delta$. Then, $\neg \forall x \varphi \in \Delta$, that $\exists x \neg \varphi \in \Delta$.

Now, Δ is witness fulfilled.

So, $\neg \varphi [y/x] \in \Delta$ for some variable y. Then, $\varphi [y/x] \notin \Delta$.

By induction hypothesis,

$M_\Delta \not\models \varphi [y/x]$, so, $M_{\Delta [x \to [y]]} \not\models \varphi$

So, $M_\Delta \not\models \forall x \varphi$. We are done

Conversely, suppose $\forall x \varphi \in \Delta$.

To show, $M_\Delta \models \forall x \varphi$. Suppose not

Then, $M_\Delta \nvDash \forall x \varphi$. Then for some term t, $M_{\Delta [x \to [t]]} \nvDash \varphi$

Let φ' be an alphabetic equivalent of φ s.t. t is substitutable for x in φ'. Then, $\forall x \varphi' \in \Delta$

[As $\vdash \varphi \Leftrightarrow \varphi'$, $\vdash \forall x \varphi \leftrightarrow \forall x \varphi'$]

Then, by the axiom,

$\forall x \varphi' \to \varphi'[t/x]$ (as t is substitutable for x in φ')

we have, $\varphi'[t/x] \in \Delta$.

Now, $M_{\Delta [x \to [t]]} \nvDash \varphi$ implies $M_{\Delta [x \to [t]]} \nvDash \varphi'$ and so, $M_\Delta \nvDash \varphi'[t/x]$ and then, by induction hypothesis, $\varphi'[t/x] \notin \Delta$. So, we arrive at a contradiction. Hence the result.

This completes the induction and proves the Truth Lemma, and thus the proof of completeness is also done.

To end with, let us write down the full axiom system for FOL.

- The propositional part.

  We have one inference rule:

  - Modus Ponens: $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

  and the following axioms:

  - $\varphi \to (\psi \to \varphi)$.
  - $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
  - $(\neg \varphi \to \psi) \to ((\neg \varphi \to \neg \psi) \to \varphi)$.

- The equality and the quantifier axioms

- $t = t$

- $(t = t') \rightarrow (\varphi(t) \leftrightarrow \varphi'(t'))$

- $\forall x \varphi \rightarrow \varphi[t/x]$, for any term t substitutable for x in φ.

- $\varphi \rightarrow \forall x \varphi$, for $x \notin FV(\varphi)$.

- $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$

We considered universalisation that was allowed anywhere in the proofs which can also be represented as inference rule :

Generalisation : $$\frac{\varphi}{\forall x \varphi}$$

H.W. Check the validity of all the axioms given above.

Theorems in FOL

1. $\vdash P_x \to \exists y\, P_y$

(a) $(\forall y \neg P_y \to \neg P_x) \to (P_x \to \neg \forall y \neg P_y)$ (Axiom 1)

(b) $\forall y \neg P_y \to \neg P_x$ (Axiom 4)

(c) $P_x \to \neg \forall y \neg P_y$ (MP on (b), (a))

(d) $P_x \to \exists y\, P_y$

2. $\vdash \exists x \forall y\, \varphi \to \forall y \exists x\, \varphi$

It is enough to show $\{\exists x \forall y\, \varphi\} \vdash \forall y \exists x\, \varphi$

[Generalisation Theorem : If $\Gamma \vdash \varphi$ and x does not occur free in Γ, then $\Gamma \vdash \forall x \varphi$]

It is enough to show $\{\exists x \forall y\, \varphi\} \vdash \exists x\, \varphi$

This is same as showing $\{\neg \forall x \neg \forall y\, \varphi\} \vdash \neg \forall x \neg \varphi$

It is enough to show $\{\forall x \neg \varphi\} \vdash \forall x \neg \forall y\, \varphi$

It is enough to show $\{\forall x \neg \varphi\} \vdash \neg \forall y\, \varphi$

It is enough to show $\{\forall x \neg \varphi, \forall y \varphi\} \vdash \bot$, that is $\{\forall x \neg \varphi, \forall y \varphi\} \vdash \psi \wedge \neg \psi$ for some formula ψ.

Let us now prove this.

1. $\forall x \neg \varphi \to \neg \varphi[x/x]$ (Axiom 4)
2. $\forall x \neg \varphi$ (Premise)
3. $\neg \varphi$ (M.P. on 2,1)
4. $\forall y \varphi \to \varphi[y/y]$ (Axiom 4)
5. $\forall y \varphi$ (Premise)
6. φ (M.P. on 5,4)
7. $\varphi \wedge \neg \varphi$ ($\psi \to (\chi \to (\psi \wedge \chi))$ is a propositional tautology)

This completes the proof.

H.W. (1) $\varphi \to \neg\neg \varphi$ in CPL.

(2) $\exists x (\varphi \wedge \psi) \to (\exists x \varphi \wedge \exists x \psi)$ in FOL.