Strategies made explicit in Dynamic Game Logic

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1 Introduction

Games play a very important role in modelling intelligent interaction. Strategies of the players playing the game form a basic ingredient of game theory, whether looked upon from the winning point of view or the best-response one, with the latter one being more relevant in the present context. The main focus of this work is to incorporate explicit notions of strategies in the framework of Dynamic Game Logic (DGL) [Par85]. Not unlike other logics talking about game and coalition structures [AHK98, Pau01], DGL suffers from '$\exists$-sickness' : the detailed level of game structures getting suppressed by existential quantifiers of "having a strategy" [Ben07a]. We intend to provide a logic ($SDGL$) that makes the game structures explicit to a great extent. A lot of other issues like the rationality of the players, their preferences, are also very important matters of concern, and we intend to include these notions in the future. This work is more in the line of some of the recent works in this area like [RS06, WHW07].

In general, strategies are partial transition relations and hence dynamic modal logic provides a good framework to talk about them, as mentioned in [Ben01, Ben02]. But the main challenge here is to combine the strategy calculus together with the game calculus. As one can easily guess, the constructs of Propositional Dynamic Logic [BdRV01] play an important role in achieving this amalgamation. In this regard, we should mention that, a lot of discussions and proposals have already been put forward by van Benthem [Benar, Ben07b]. This effort can be looked upon as a follow-up of one of these proposals.

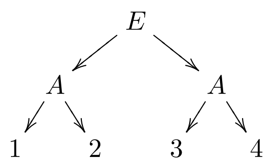
After providing a brief overview of DGL in the next section, we propose a logic for strategizing DGL ($SDGL$) in section 3 with a complete axiomatization. Section 4 provides some ideas for further work.

2 Dynamic Game Logics : an overview

We now give a brief review of DGL, the dynamic game logic of two-person sequential games in this section, which was first proposed in [Par85], and further developed by [Pau01], [Ben03], [BGL07] and others. DGL talks about 'generic' games which can be played starting from any state s on the 'game boards' and the semantics is based on the 'forcing relations' describing the powers each player has to end in a set of final states, starting from a single initial state.

> $s\rho_G^i X$: player i has a *strategy* for playing game G from state s onwards, whose resulting final states are always in the set X, whatever the other players choose to do.

To exemplify, let us move onto real extensive games for once. Consider the game tree:



In this game, player E has two strategies, forcing the sets of end states $\{1, 2\}$, $\{3, 4\}$, while player A has four strategies, forcing one of the sets of states $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$.

These forcing relations satisfy the following two simple set-theoretic conditions [Ben99]:

(C1) Monotonicity: If $s\rho_G^i X$ and $X \subseteq X'$, then $s\rho_G^i X'$.

(C2) Consistency: If $s\rho_G^E Y$ and $s\rho_G^A Z$, then Y and Z overlap.

In the semantics of DGL as proposed in [Par85, Pau01], another extra condition is assumed :

(C3) Determinacy: If it is not the case that $s\rho_G^E X$, then, $s\rho_G^A S\text{-}X$, and the same for A vis-a-vis E.

Both [Par85] and [Pau01] talks about determined games. This simplifies things a lot, but fails to express the roles of the players in the games. The dynamic logic for non-determined games was studied extensively in [BGL07] which also introduced the notion of parallel games in the syntax. For the present work the concurrent game construct has not been dealt with. The *iteration* operation for repeated play as present in [Par85] has also not been considered here. We only consider the following constructs which form new games: *choice* $(G \cup G')$, *dual* (G^d), *and sequential composition* $(G; G')$. The readers could easily guess the intuitive meanings of these constructs. For the sake of continuation to the next section, in what follows, the DGL for non-determined games has been briefly discussed. To start with, it should be noted here that the players' powers have a recursive structure in the complex games:

Fact 2.1 *Forcing relations for players in complex sequential two-person games satisfy the following equivalences:*

$$
\begin{array}{ccc}
s\rho^E_{G\cup G'} X & \text{iff} & s\rho^E_G X \text{ or } s\rho^E_{G'} X \\
s\rho^A_{G\cup G'} X & \text{iff} & s\rho^A_G X \text{ and } s\rho^A_{G'} X \\
s\rho^E_{G^d} X & \text{iff} & s\rho^A_G X \\
s\rho^A_{G^d} X & \text{iff} & s\rho^E_G X \\
s\rho^i_{G;G'} X & \text{iff} & \exists Z : s\rho^i_G Z \text{ and for all } z \in Z,\ z\rho^i_{G'} X.
\end{array}
$$

The basic models that play the role of game boards are as follows:

Definition 2.2 *A game model is a structure $\mathcal{M} = (S, \{\rho^i_g \mid g \in \Gamma\}, V)$, where S is a set of states, V is a valuation assigning truth values to atomic propositions in states, and for each $g \in \Gamma$, $\rho^i_g \subseteq S \times \mathcal{P}(S)$. We assume that for each g, the relations are upward closed under supersets (the earlier Monotonicity), while also, the earlier Consistency condition holds for the forcing relations of the players A, E.*

The language of DGL (without game iteration) is defined as follows:

Definition 2.3 *Given a set of atomic games Γ and a set of atomic propositions Φ, game terms γ and formulas ϕ are defined inductively:*

$$
\begin{array}{l}
\gamma := g \mid \phi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d \\
\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle \gamma, i\rangle \phi,
\end{array}
$$

where $p \in \Phi$, $g \in \Gamma$ and $i \in \{A, E\}$.

The truth definition for formulas ϕ in a model $\mathcal{M}$ at a state s is standard, except for the modality $\langle \gamma, i\rangle\phi$, which is interpreted as follows:

$\mathcal{M}, s \models \langle \gamma, i\rangle\phi$ iff there exists $X : s\rho^i_\gamma X$ and $\forall x \in X : \mathcal{M}, x \models \phi$.

The complete axiomatization of this logic has been proved in [BGL07] :

Theorem 2.4 *DGL is complete and its validities are axiomatized by the following axioms:*

a) all propositional tautologies and inference rules

b) if $\vdash \phi \rightarrow \psi$ then $\vdash \langle g, i\rangle\phi \rightarrow \langle g, i\rangle\psi$

c) $\langle g, E\rangle\phi \rightarrow \neg\langle g, A\rangle\neg\phi$

d) reduction axioms:

$$\langle \alpha \cup \beta, E\rangle\phi \leftrightarrow \langle \alpha, E\rangle\phi \vee \langle \beta, E\rangle\phi$$
$$\langle \alpha \cup \beta, A\rangle\phi \leftrightarrow \langle \alpha, A\rangle\phi \wedge \langle \beta, A\rangle\phi$$

$$\langle \gamma^d, E\rangle\phi \leftrightarrow \langle \gamma, A\rangle\phi$$
$$\langle \gamma^d, A\rangle\phi \leftrightarrow \langle \gamma, E\rangle\phi$$
$$\langle \alpha;\beta, i\rangle\phi \leftrightarrow \langle \alpha, i\rangle\langle \beta, i\rangle\phi$$
$$\langle \delta?, E\rangle\phi \leftrightarrow (\delta \wedge \phi)$$
$$\langle \delta?, A\rangle\phi \leftrightarrow (\neg\delta \wedge \phi).$$

This logic is also decidable. As can be noticed, the truth definition of the modal game formulas of the form $\langle \gamma, i\rangle\phi$ is given in terms of existence of strategies, without going into their structures. In what follows, the strategy structures have been explicitly dealt together with the game structures.

3 Strategizing DGL

3.1 A logic for strategies

Mentioning strategies explicitly in the dynamic game logic framework prompt us to divert from the usual DGL semantics that takes into consideration 'generic' games on game boards. The whole point is to bring *strategies* within the logical language which till now have their place in giving meaning to the game as well as coalition modalities [Pau01].

Adding explicit strategy terms to DGL, the language of Strategized DGL ($SDGL$) is defined by,

Definition 3.1 *Given a set of atomic games* Γ, *a set of atomic strategies* Σ, *a set of atomic actions* Π *and a set of atomic propositions* Φ, *game terms* γ, *strategy terms* σ, *action terms* π *and formulas* ϕ *are defined inductively as follows:*

$$\gamma := g \mid \phi? \mid \gamma^d \mid \gamma;\gamma \mid \gamma \cup \gamma$$
$$\sigma := s \mid \sigma \cup \sigma \mid \sigma;\sigma$$
$$\pi := b \mid \pi \cup \pi \mid \pi^*$$
$$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid [\pi]\phi \mid \langle\pi\rangle\phi \mid \langle \sigma, i, \gamma\rangle\phi$$

where $p \in \Phi$, $s \in \Sigma$, $g \in \Gamma$, $b \in \Pi$, Π *being a finite set, and* $i \in \{A, E\}$.

Regarding the intuitive understanding of the strategy terms, '$\cup$' corresponds to choice of strategies, and ';' to composition of strategies. It should be mentioned here that, the way the semantics is given later, it would have been enough to use just one combination operation of the strategy terms. The use of both of them aids in the intuitive understanding.

Moving away from the 'generic' game structures, the models take the form of extensive game trees with a few additional actions. Before going into all these, we need a parent model which is given as follows.

Definition 3.2 *A model is a structure* $\mathcal{M} = \langle$ *S*, $\{R_\pi : \pi$ *'s are actions*$\}$, ref, **L**, **R**, *V* $\rangle$, *where S is a set of states and V is a valuation assigning truth values to atomic propositions in states. For each* π, R_π *is a binary relation on S.* ref, **L**, **R** *are all reflexive relations over S, together with* $\langle$ *S*, $\{R_\pi : \pi$ *'s are actions*$\}\rangle$ *forming a regular action frame.*

In this model, atomic and composite games from a specified 'start'-state are defined in the following. It should be mentioned that all these game structures are taken to be finite, defined over finite subsets of S.

Definition 3.3 *Game(*$\mathcal{M}$*, s,* γ*) is a structure defined recursively as follows:*

(i) For atomic games g, Game($\mathcal{M}$*, s, g) is a structure given by,*
$\langle W \subseteq S, s \in W, \{R_b \downarrow_W : b \in \Pi\}, V = V^{\mathcal{M}} \downarrow_W, P : W \to \{E, A, end\}\rangle$.

(ii) For test games $\phi?$*, Game(*$\mathcal{M}$*, s,* $\phi?$*) is a structure given by,*
$\langle \{s\}, s, ref \downarrow_{\{s\}}, V = V^{\mathcal{M}} \downarrow_{\{s\}}, P : \{s\} \to \{end\}\rangle$.

(iii) Given Game($\mathcal{M}$, s, γ), Game($\mathcal{M}$, s, γ^d) is the structure $\langle W \subseteq S, s \in W, \{R_b\downarrow_W : b \in \Pi\}, V = V^{\mathcal{M}}\downarrow_W, P : W \to \{E, A, end\}\rangle$, where all the constituents of the structure are the same as the corresponding ones in Game($\mathcal{M}$, s, γ), except for P_{γ^d}, which satisfies the property: $P_{\gamma^d}(w)$ = E/A, whenever $P_\gamma(w)$ = A/E, respectively.

(iv) Given Game($\mathcal{M}$, s, γ) and Game($\mathcal{M}$, s, γ'), Game($\mathcal{M}$, s, $\gamma \cup \gamma'$) is the structure $\langle W \subseteq S, s \in W, \{R_b\downarrow_W : b \in \Pi\}, \mathbf{L}\downarrow_{\{s,s\}}, \mathbf{R}\downarrow_{\{s,s\}}, V = V^{\mathcal{M}}\downarrow_W, P : W \to \{E, A, end\}\rangle$, where $W = W_\gamma \uplus W_{\gamma'}$, and P extends both P_γ and $P_{\gamma'}$.

(v) Given Game($\mathcal{M}$, s_1, γ) and Game($\mathcal{M}$, s_2, γ'), Game($\mathcal{M}$, s, $\gamma;\gamma'$) is defined if for each $t \in P_\gamma^{-1}(end)$, Game($\mathcal{M}$, t, γ') can be defined. Suppose we have Game($\mathcal{M}$, t_1, γ'), ..., Game($\mathcal{M}$, t_n, γ'). In that case, Game($\mathcal{M}$, s, $\gamma;\gamma'$) is the structure $\langle W \subseteq S, s \in W, \{R_b\downarrow_W : b \in \Pi\}, V = V^{\mathcal{M}}\downarrow_W, P : W \to \{E, A, end\}\rangle$, where $W = W_\gamma \cup W^1_{\gamma'} \cup \ldots \cup W^n_{\gamma'}$; $s = s_1$; P extends P_γ, P^1_γ, ..., P^n_γ, with the restriction that for $w \in P_\gamma^{-1}(end) \cap W$, $P(w) = P_{\gamma'}(s_2)$.

Because of some technical reasons regarding satisfiability, *choice* games can only be defined for the games with the same initial state, which is not really a big issue. The sequential composition game could also be defined under certain restrictions as mentioned above. It is now time to define strategies of the players in a game, which again has a recursive definition. Note that we will only talk about full strategies here and the definition is given likewise.

Definition 3.4 *Given Game($\mathcal{M}$, s, γ), a strategy for a player i, given by the relation $\mathcal{R}_i^\gamma$ is defined by,*

(i) For Game($\mathcal{M}$, s, g), $\mathcal{R}_E^g[\mathcal{R}_A^g] \subseteq \bigcup\{R_b\downarrow_{W_g} : b \in \Pi\}$ satisfying the following conditions:

(a) $s \in Dom(\mathcal{R}_E^g)[Dom(\mathcal{R}_A^g)]$, and $Ran(\mathcal{R}_E^g)[Ran(\mathcal{R}_A^g)] \cap P_g^{-1}(end) \neq \emptyset$

(b) For each $t \in P_g^{-1}(E, A) - \{s\}$, $t \in Dom(\mathcal{R}_E^g)[Dom(\mathcal{R}_A^g)]$ iff $t \in Ran(\mathcal{R}_E^g)[Ran(\mathcal{R}_A^g)]$.

(c) For each $s \in P^{-1}(E)[P^{-1}(A)]$, $\exists$ unique s' such that $(s, s') \in \mathcal{R}_E^g[\mathcal{R}_A^g]$.

(d) For each $s \in P^{-1}(A)[P^{-1}(E)]$, $(s, s') \in \bigcup\{R_b\downarrow_{W_g} : b \in \Pi\}$ implies $(s, s') \in \mathcal{R}_E^g[\mathcal{R}_A^g]$.

(e) Nothing else is in $\mathcal{R}_E^g[\mathcal{R}_A^g]$.

(ii) For Game($\mathcal{M}$, s, $\phi?$), $\mathcal{R}_i^{\phi?}$ = ref$\downarrow_{\{s\}}$.

(iii) For Game($\mathcal{M}$, s, γ^d), $\mathcal{R}_E^{\gamma^d} = \mathcal{R}_A^\gamma$, and $\mathcal{R}_A^{\gamma^d} = \mathcal{R}_E^\gamma$.

(iv) For Game($\mathcal{M}$, s, $\gamma \cup \gamma'$), $\mathcal{R}_E^{\gamma\cup\gamma'} = \mathbf{L}\downarrow_{\{s,s\}} \cup \mathcal{R}_E^\gamma$ or, $\mathbf{R}\downarrow_{\{s,s\}} \cup \mathcal{R}_E^{\gamma'}$,
and, $\mathcal{R}_A^{\gamma\cup\gamma'} = \mathbf{L}\downarrow_{\{s,s\}} \cup \mathbf{R}\downarrow_{\{s,s\}} \cup \mathcal{R}_A^\gamma \cup \mathcal{R}_A^{\gamma'}$.

(v) For Game($\mathcal{M}$, s, $\gamma;\gamma'$), $\mathcal{R}_i^{\gamma;\gamma'} = \mathcal{R}_i^\gamma \cup \mathcal{R}_{i_{j_1}}^{\gamma'} \cup \ldots \cup \mathcal{R}_{i_{j_l}}^{\gamma'}$, where the indices correspond to the number of times the 'end'-state is reached in $\mathcal{R}_i^\gamma$.

For an example of the players' strategies, consider the simple extensive game tree:

$$\begin{tikzcd}[column sep=small]
 & & & s_1, E \arrow[dll] \arrow[drr] & & & \\
 & s_2, A \arrow[dl] \arrow[dr] & & & & s_3, A \arrow[dl] \arrow[dr] & \\
s_4, end & & s_5, end & & s_6, end & & s_7, end
\end{tikzcd}$$

The strategies of E are $\{(s_1, s_2), (s_2, s_4), (s_2, s_5)\}$, and $\{(s_1, s_3), (s_3, s_6), (s_3, s_7)\}$, whereas the strategies for A are $\{(s_1, s_2), (s_1, s_3), (s_2, s_4), (s_3, s_6)\}$, $\{(s_1, s_2), (s_1, s_3), (s_2, s_4), (s_3, s_7)\}$, $\{(s_1, s_2), (s_1, s_3), (s_2, s_5), (s_3, s_6)\}$, and $\{(s_1, s_2), (s_1, s_3), (s_2, s_5), (s_3, s_7)\}$.

If we consider the *choice* operations of two such games, the strategies of the players could be easily computed. For the *sequential composition*, consider the following two games:

$$\begin{tikzcd}[column sep=small]
 & s_1, E \arrow[dl] \arrow[dr] & & & & t_1, A \arrow[dl] \arrow[dr] & \\
s_2, end & \mathbf{G} & s_3, end & & t_2, end & \mathbf{H} & t_3, end
\end{tikzcd}$$

The strategies of E in G are $\{(s_1, s_2)\}$, and $\{(s_1, s_3)\}$, and in H is $\{(t_1, t_2), (t_1, t_3)\}$, and similarly, that for A in G is $\{(t_1, t_2), (t_1, t_3)\}$, and in H are $\{(t_1, t_2)\}$, and $\{(t_1, t_3)\}$.

Suppose, the model is such that $G; H$ could be defined and it is as follows:

$$\begin{tikzcd}[column sep=small]
 & & & s_1, E \arrow[dll] \arrow[drr] & & & \\
 & s_2, A \arrow[dl] \arrow[dr] & & & & s_3, A \arrow[dl] \arrow[dr] & \\
t_4, end & & t_5, end & & t_6, end & & t_7, end
\end{tikzcd}$$

The readers can notice that it is just the game given as example earlier, and hence could easily verify that the strategies of the players in this complex game conform with the definition given to compute the strategies of the *sequential composition* games, from the simpler ones.

Before going into the truth-definitions of formulas, let us mention a few words about interpreting the strategy terms of the language. The strategy terms are always interpreted corresponding to some game structure Game($\mathcal{M}$, s, γ) and player i. Let $\mathbb{R}_i^\gamma$ denote the set of all strategies for player i in Game($\mathcal{M}$, s, γ).

Definition 3.5 *Given Game($\mathcal{M}$, s, γ) and player i, a strategy function $\mathcal{F}_i^\gamma$ is a partial function from the set of all strategy terms to $\mathbb{R}_i^\gamma$, satisfying the following conditions.*

(i) For $s \in \Sigma$, $\mathcal{F}_i^\gamma(s)$ is defined, only when γ is an atomic or a test game.

(ii) For the choice game $\alpha \cup \beta$, $\mathcal{F}_i^{\alpha\cup\beta}$ is given by,

$$\begin{array}{rcl}
\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{L}\downarrow_{W_{\alpha\cup\beta}} \cup \mathcal{R}_E^\alpha & \textit{iff} & \mathcal{F}_E^\alpha(\sigma) = \mathcal{R}_E^\alpha, \\
\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{R}\downarrow_{W_{\alpha\cup\beta}} \cup \mathcal{R}_E^\beta & \textit{iff} & \mathcal{F}_E^\beta(\tau) = \mathcal{R}_E^\beta, \\
\mathcal{F}_A^{\alpha\cup\beta}(\sigma \cup \tau) = \mathcal{R}_A^{\alpha\cup\beta} & \textit{iff} & \mathcal{F}_A^\alpha(\sigma) = \mathcal{R}_A^\alpha \\
 & & \textit{and, } \mathcal{F}_A^\beta(\tau) = \mathcal{R}_A^\beta.
\end{array}$$

(iii) $\mathcal{F}_E^{\gamma^d}(\sigma) = \mathcal{R}_E^{\gamma^d}$ *iff* $\mathcal{F}_A^\gamma(\sigma) = \mathcal{R}_A^\gamma$, $\mathcal{F}_A^{\gamma^d}(\sigma) = \mathcal{R}_E^{\gamma^d}$ *iff* $\mathcal{F}_A^\gamma(\sigma) = \mathcal{R}_A^\gamma$.

(iv) For the composition game $\alpha; \beta$, $\mathcal{F}_i^{\alpha;\beta}$ satisfies,

$$\mathcal{F}_i^{\alpha;\beta}(\tau; \eta) = \mathcal{R}_i^{\alpha;\beta} \quad \textit{iff} \quad \mathcal{F}_i^\alpha(\tau) = \mathcal{R}_i^\alpha \textit{ and } \mathcal{F}_i^\beta(\eta) = \mathcal{R}_i^\beta.$$

Note that the way these partial functions are given, it takes care of the cases of mismatched syntax (like, $\langle \sigma \cup \tau, E, \alpha; \beta\rangle\phi$), which does not have any corresponding structure in the model. For the semantics of our language, we define the truth of a formula ϕ in $\mathcal{M}$ at a state s in the obvious manner, with the action modalities defined in the usual PDL-style and the following key clause for the game-strategy modality:

$$\mathcal{M}, s \models \langle \sigma, i, \gamma\rangle\phi \text{ iff for all } s' \in Ran(\mathcal{F}_i^\gamma(\sigma)) \cap P^{-1}(end) \text{ in Game}(\mathcal{M}, s, \gamma), \mathcal{M}, s' \models \phi.$$

Here are some validities of this logic.

- $\langle \sigma, i, \gamma\rangle\phi \to \langle \sigma, i, \gamma\rangle(\phi \vee \psi)$
- $\langle \sigma, i, \gamma\rangle(\phi \wedge \psi) \leftrightarrow \langle \sigma, i, \gamma\rangle\phi \wedge \langle \sigma, i, \gamma\rangle\psi$

3.2 Axioms and completeness

We now provide a complete axiomatization of $SDGL$.

Theorem 3.6 *$SDGL$ is complete and its validities are axiomatized by*

a) all propositional tautologies and inference rules

b) generalization rule for the action modalities *c) axioms for the action constructs:*

$$[\pi](\phi \to \psi) \to ([\pi]\phi \to [\pi]\psi)$$
$$\langle\pi\rangle\phi \leftrightarrow \neg[\pi]\neg\phi$$
$$\langle\pi_1 \cup \pi_2\rangle\phi \leftrightarrow \langle\pi_1\rangle\phi \vee \langle\pi_2\rangle\phi$$
$$\langle\pi^*\rangle\phi \leftrightarrow (\phi \vee \langle\pi\rangle\langle\pi^*\rangle\phi)$$
$$[\pi^*](\phi \to [\pi]\phi) \to (\phi \to [\pi^*]\phi)$$

d) $\langle s, i, g\rangle\phi \to \langle b_1 \cup \ldots \cup b_n\rangle\langle(b_1 \cup \ldots \cup b_n)^*\rangle\phi$*, where* $\Pi = \{b_1, \ldots, b_n\}$

e) $\langle\sigma, i, \gamma\rangle(\phi \to \psi) \to (\langle\sigma, i, \gamma\rangle\phi \to \langle\sigma, i, \gamma\rangle\psi)$

f) if $\vdash \phi \to \psi$ *then* $\vdash \langle\sigma, i, \gamma\rangle\phi \to \langle\sigma, i, \gamma\rangle\psi$

g) reduction axioms:

$$\langle\sigma \cup \tau, E, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, E, \alpha\rangle\phi \vee \langle\tau, E, \beta\rangle\phi$$
$$\langle\sigma \cup \tau, A, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, A, \alpha\rangle\phi \wedge \langle\tau, A, \beta\rangle\phi$$
$$\langle\sigma, E, \gamma^d\rangle\phi \leftrightarrow \langle\sigma, A, \gamma\rangle\phi$$
$$\langle\sigma, A, \gamma^d\rangle\phi \leftrightarrow \langle\sigma, E, \gamma\rangle\phi$$
$$\langle\tau; \eta, i, \alpha; \beta\rangle\phi \leftrightarrow \langle\tau, i, \alpha\rangle\langle\eta, i, \beta\rangle\phi$$
$$\langle\sigma, E, \delta?\rangle\phi \leftrightarrow (\delta \wedge \phi)$$
$$\langle\sigma, A, \delta?\rangle\phi \leftrightarrow (\neg\delta \wedge \phi)$$

h) strategy rules:

for each $X \subseteq \Pi$*, the rule below:*

if $\vdash \phi \to \langle(\cup X)\rangle\langle(\cup X)^*\rangle\psi$ *then* $\vdash \phi \to \langle s, i, g\rangle\psi$.

Soundness of these axioms follow straightforwardly from the definition of the game structures and the strategies, as well as the strategy functions. The completeness proof can be summarized using the following two insights. First, the reduction axioms for the powers of the players' strategies turn every formula into an equivalent one involving only modalities for atomic game and strategy terms. Secondly, the axiom and the rules connecting the game-strategy modality with the action modality provide a way for describing the strategies in the overall game structures through the intermediate moves made by the players.

4 Further work

DGL talks about *generic* games played on game boards, and the meaning of the game modalities is given by existence of strategies. $SDGL$ brings out these strategies to the fore. It is clear that there are some sentences which could be expressed in $SDGL$, but not in DGL. But it is also the case that there are certain statements that can be expressed in DGL, but not in $SDGL$: for example, 'player i does not have any strategy in the game g to achieve ϕ' can be expressed in DGL as $\neg\langle g, i\rangle\phi$. So it would be ideal to have a logic that could express both. This gives rise to the following issue:

Question What would be the complete axiomatization of a logic that has both Parikh's original game modalities as well as the game-strategy modalities presented in the earlier section?

Several other languages talking about game structures and coalition structures like Alternating-time temporal logic and Coalition logic could also be investigated so as to add an explicit notion of strategies, which merely occur as an existential notion in the semantics of these logics. This could very well aid in the social choice mechanism designs. Epistemic versions of these game logics with explicit strategies, where one may incorporate knowledge/belief as well as preference modalities is also an interesting area for investigation which could be explored in great details.

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