A qualitative approach to uncertainty

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Abstract. We focus on modelling dual epistemic attitudes (belief-disbelief, knowledge-ignorance, like-dislike) of an agent. This provides an interesting way to express different levels of uncertainties explicitly in the logical language. After introducing a *dual* modal framework, we discuss the different possibilities of an agent's attitude towards a proposition that can be expressed in this framework, and provide a preliminary look at the dynamics of the situation.

1 Introduction

Real life is not boolean. Many of the situations, events and concepts that we face in our everyday life cannot be classified in a simple binary hierarchy. When we wake up, the morning may not be dark with thunder clouds hovering, but it may not be sunny too (many-valuedness); we believe that our favorite team will win tonight's soccer match, but we still consider the possibility of a loss or even a tie (modality); the dice which is about to be cast will result in any of the six different outcomes (probabilistic); the director of the company seems to be young, but we are not sure (fuzziness). Thus, uncertainty arises in various forms and various connotations. What we will deal with here is not vagueness or impreciseness of concepts (which leads to fuzzy logic, many-valued logic; see [1]), but rather uncertainties in agent's attitudes (e.g. beliefs).

The main aim of this paper is to deal with negative attitudes of agents on a par with their positive attitudes; this will allow us to describe different levels of uncertainties in a more classical framework. Though there are previous works modelling these dual attitudes (e.g. [2,3]), the novelty of this paper lies in the fact that we are expressing different levels of agent uncertainty explicitly in the logical language.

We propose a bi-modal framework that allows us to express various kinds of attitudes toward a formula φ. Our work is based on the following observations. First, the modal operator $\Box$ allows us to express what is true in all worlds reachable by the accessibility relation. Such propositions, which are true in those

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accessible worlds, may be false elsewhere. The worlds where the propositions are false should be considered, if we want to deal with attitudes like disbelief, dislike or ignorance of agents. Secondly, every interpretation of the accessibility relation (knowledge, belief, like) has its *dual* concept (ignorance, disbelief, dislike, respectively), which is usually not represented in the framework[3]. The reason that they are not expressed in the existing literature is that, usually it is assumed that these two concepts are complementary (that is, representable by negations). In general, this does not have to be the case: the fact that we dislike the rain does not imply that we like when it is not raining. On the other hand, suppose we believe that a horse will win a race if its rating is above certain number, and disbelieve it if its below some other number. "Disbelieving" will not equate to "not believing" in this case.

Let us consider a fair 100 ticket lottery. Though we do believe that exactly one of the tickets will win the lottery, we have doubts regarding the possibility of an individual ticket winning it. One can resolve this paradoxical situation by replacing the classical negation (believing that the ticket number 99 will not win) by the weaker notion of disbelief (disbelieving that the ticket number 99 will win). This applies to many practical situations as well. A crime has been committed and two of your very good friends are the prime suspects. It is really hard for you to believe that any one of them has committed the crime, yet the circumstantial evidence forces you to believe that either of them did it.

From the perspective of belief merging, consider k sources of information providing their opinions regarding a certain event p. Suppose that m of them state that p holds, and n of them state that p does not hold. Some of the sources may not have any opinion regarding p, but none of the sources are inconsistent in the sense that they do not simultaneously state that p holds and does not hold. So we have that $m + n \leq k$. The fraction m/k can be seen as the degree of certainty of the source that p holds and n/k that ¬p holds. Let $cr(p) \in [0, 1]$ denote the degree of certainty that p holds. We can think of threshold values t_1 and t_2 $(0 < t_1 \leq t_2 \leq 1)$ for belief and disbelief, that is, p is believed if $t_2 \leq cr(p)$, and is disbelieved if $cr(p) < t_1$. In the remaining cases p is neither believed nor disbelieved. So, from the fact the p is not believed, we cannot say that p is disbelieved. Also, from the fact that p is disbelieved, it does not necessarily follow that $\neg p$ is believed. [4]

As exemplified above, considering the dual of positive notions of knowledge, belief and others are important in their own right. Moreover, this also opens the door for modeling different kinds of uncertain attitudes that arise, when the agent is placed in decision making scenarios.

Consider the following decision-making scenario. Suppose Alice has to elect a member of parliament from her constituency for the next five year term. Several candidates are in the fray, say, A, B, C, D, E. Generally, these situations are modelled by the notion of *preference*. But we feel that, while considering the agent's preference over candidates, the intricate feelings of uncertainties that

[3] Here we do not refer to the dual of the modal operator $\Box$, which is $\Diamond$.

[4] These examples are taken from [2,4].

the agent may have regarding the candidates get lost. Let I denote that I is a good candidate for the task at hand. Alice may be undecided about A and B, has strong positive opinion about C, negative opinion about D, and have not even heard of E. By introducing the dual framework (cf. Section 3), we will be able to model all the intricate levels of Alice's uncertainties.

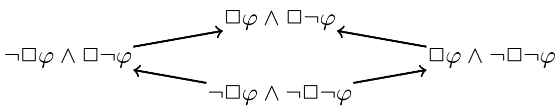
2 Some relevant issues: a prelude

In this section, we discuss different issues leading up to our proposal. We start by providing a systematic sketch about the attitudes expressible by a single modal operator. Then we give a brief overview of the relevant literature dealing with positive and negative sides of information, and finally, we provide a brief sketch about various approaches to uncertainty.

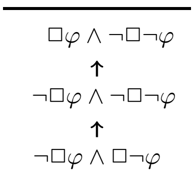
2.1 Attitudes representable with a single modal operator

Modal logic, one of the simplest intensional logics, allows us to express attitudes by means of the modal operator $\Box$. Formulas of its language are built from a set of atomic propositions $\mathtt{P}$ by closing it under negation $\neg$, conjunction $\wedge$ and the modal operator $\Box$. Formulas of the form $\Box\varphi$ can be read as *"the agent has a positive attitude towards φ"*, and $\Diamond\varphi$ is an abbreviation for $\neg\Box\neg\varphi$.

Such formulas are evaluated in Kripke models. Formulas of the form $\Box\varphi$ and $\Box\neg\varphi$ can be true or false, giving us four possible attitudes towards φ. These possibilities can be ordered according to the amount of information about φ each one provides, that is, the number of true formulas in the set $\{\Box\varphi, \Box\neg\varphi\}$. This yields the power set of a 2-element set ordered by inclusion:



The case with $\Box\varphi$ and $\Box\neg\varphi$ is undesirable under some interpretations (eg. knowledge) of $\Box$ since it represents having positive attitude towards both a formula and its negation (what we will call $\Box$-inconsistency). To avoid this case, it is usually asked for the accessibility relation in the Kripke model to be at least *serial* (in knowledge interpretations, the stronger *reflexivity* is required), and we are left with three possible attitudes which can also be ordered according to the attitude of the agent about φ: from positive towards $\neg\varphi$ to positive attitude about φ (see the diagram below).



Two of these three $\Box$-consistent cases express certainty (towards φ or $\neg\varphi$), and we are left with just one attitude to express uncertainty. This is not enough to differentiate between say, "*not interested and having no opinion*" or "*interested but indecisive*" about φ.

2.2 Bipolar representation of information

Prevalent modal approaches that deal with information and attitudes mostly focus on representing only the positive information an agent has about a subject. But there is also a complementary view: just as we have positive information about a subject, we can also have (independent) negative information about it.

The idea of representing both positive and negative aspects of a subject is not new. There are approaches with such proposals in many areas, like decision theory [5], argumentation theory [6], and many others (see [7] for an overview). The concept of *bipolarity* is precisely about this: an explicit handling of the positive and negative aspects in information [8]. It is based on the fact that, when taking a decision or weighing some possibilities, we consider not only the positive aspects of the available options, but also the negative ones.

From this perspective, frameworks that consider only the positive aspect can be seen as special situations in which the positive and the negative information are mutually exclusive and mirror images of each other: I consider p as good if and only if I consider $\neg p$ as bad. But this does not need to be the case: we can imagine a situation in which, though p is good, its negation $\neg p$ is not necessarily bad, and the notion of bipolarity allows us to deal with such cases. These are precisely the kind of situations that we are interested in, and the framework presented in Section 3 allows us to deal with them.

2.3 Approaches for modeling uncertainty

Considering such dual frameworks for positive and negative information paves the way for an in-depth study of qualitative representation of uncertainty (cf. Section 3). We now give a brief overview of the highly active research area of uncertainty-modeling. The later half of the past century witnessed several proposals for modeling uncertainty, with a focus to formalize human (commonsense reasoning). To mention a few of the relevant approaches, we can refer to fuzzy set theory [9], possibility theory [10], rough set theory [11], probabilistic approaches [12], and Dempster-Shafer theory [13,14]. These comprise quantitative ways of dealing with uncertainties, but there have been some qualitative approaches based on ordered sets [12]. Different kinds of propositional and first-order frameworks have also been proposed describing different interpretations of uncertainty, e.g. probability logics [15], fuzzy logics [16], possibilistic logics, [17], rough logics [18], and many-valued logics [19].

Evidently, there are varied approaches to deal with uncertainties, but mathematical structures play a very important role in these formulations. The logical languages describing the uncertainty concept generally remain the same, only their interpretations change in the different theories. Some of these theories are more suitable for describing vague or uncertain concepts, while others are more appropriate for describing mental attitudes of agents. Our interests lie in describing diverse mental attitudes of agents *more explicitly in the logical language.* To achieve such a goal, the following section presents a special kind of dual framework, viz. a logical language talking about positive and negative attitudes of agents regarding events. This will give us a way to talk about uncertainties of agents in a bi-modal framework.

3 A language for positive and negative attitudes

We now introduce an extension of the classical modal language that allows us to express both positive and negative attitudes *explicitly*. After presenting the language, models, and semantic interpretation, we show how, even by imposing strong *consistency* requirements, we still can express more attitudes than the classical modal framework.

3.1 The basic system

Definition 3.1. *Let* $\mathtt{P}$ *be a set of atomic propositions. Formulas* φ *of the language* $\mathcal{L}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [+]\varphi \mid [-]\varphi$$

with $p \in \mathtt{P}$*. Formulas of the form* $[+]\varphi$ *(*$[-]\varphi$*) are read as "the agent has a positive (negative) attitude towards* φ*". The corresponding 'diamond' modalities are defined as usual.*

Having both the positive and the negative attitudes represented explicitly allow us to express combinations of them. For example, in a discussion about preferences, the notions can be read as *like* and *dislike*, and then we can express situations like *the agent is undecided whether she likes (that is, she likes and dislikes) rainy day* ($[+]rd \wedge [-]rd$). In a doxastic context the notions can be read as *belief* and *disbelief*, and we can express ideas like *the agent does not have any opinion (neither believes nor disbelieves) regarding whether it will rain tomorrow* ($\neg[+]rt \wedge \neg[-]rt$). We can even interpret them in terms of knowledge, and express combinations of *knowledge* and *ignorance*.

Definition 3.2 (Dual model). *Given a set of atomic propositions* $\mathtt{P}$*, a* dual model *is a tuple* $\mathcal{M} = \langle W, R^+, R^-, V\rangle$ *where* W *is a non-empty set of worlds,* R^+ *and* R^- *are binary relations on* W *and* $V : \mathtt{P} \to \wp(W)$ *is a valuation function. We denote by* $\boldsymbol{M}$ *the class of all semantic models.*

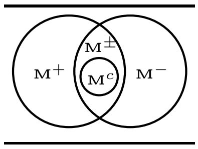
The difference between our system and an ordinary bi-modal framework relies on the interpretation of negative attitude formulas $[-]\varphi$.

Definition 3.3. *Let* $\mathcal{M} = \langle W, R^+, R^-, V\rangle$ *be a dual semantic model and let* w *be a world in it. Atomic propositions, negation and conjunction are interpreted as usual. For the modalities, we have*

$$(\mathcal{M}, w) \models [+]\varphi \quad \text{iff} \quad \text{for all } w' \text{ such that } R^+ww',\ (\mathcal{M}, w') \models \varphi$$
$$(\mathcal{M}, w) \models [-]\varphi \quad \text{iff} \quad \text{for all } w' \text{ such that } R^-ww',\ (\mathcal{M}, w') \models \neg\varphi.$$

Among the class of all dual semantic models, we distinguish four of them.

Definition 3.4. *A dual model $\mathcal{M}$ is in*



- *the class of* $[+]$-consistent *models* ($\mathbf{M}^+$) *iff* R^+ *is* serial*;*
- *the class of* $[-]$-consistent *models* ($\mathbf{M}^-$) *iff* R^- *is* serial*;*
- *the class of* $[\pm]$-consistent *models* ($\mathbf{M}^\pm$) *iff* R^+ *and* R^- *are* serial*;*
- *the class of* dual-consistent *models* ($\mathbf{M}^c$) *iff for every* $w \in W$ *we have* $R^+[w] \cap R^-[w] \neq \emptyset$.

It can be easily verified that in $\mathbf{M}^+$-models, the formula $[+](\varphi \wedge \neg\varphi)$ is not satisfiable, hence the name of the class. Similarly, $[-](\varphi \vee \neg\varphi)$ is not satisfiable in $\mathbf{M}^-$-models, and both formulas are not satisfiable in $\mathbf{M}^\pm$-models. The class $\mathbf{M}^c$ is more restrictive than $\mathbf{M}^\pm$ since, besides having both R^+ and R^- serial, their intersection should be non-empty. In models of such class, the two mentioned formulas are not satisfiable, and so is $[+]\varphi \wedge [-]\varphi$.

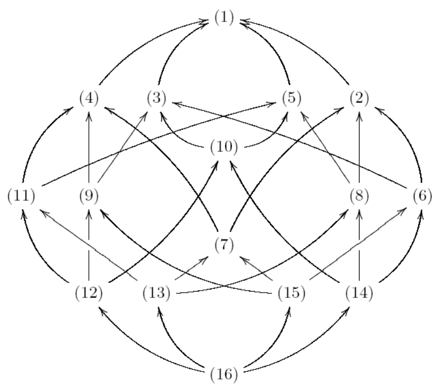
3.2 The new attitudes

In a language with a single normal modality we can only express four attitudes with respect to any given formula φ, as discussed earlier. With our new modality $[-]$ we have sixteen combinations of truth-values for the formulas $[+]\varphi$, $[+]\neg\varphi$, $[-]\varphi$ and $[-]\neg\varphi$, all of them satisfiable in $\mathbf{M}$. To make a systematic analysis, we consider all the cases.

1: $\big([+]\varphi \wedge [+]\neg\varphi\big) \wedge \big([-]\varphi \wedge [-]\neg\varphi\big)$	2: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big([-]\varphi \wedge [-]\neg\varphi\big)$
3: $\big([+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big([-]\varphi \wedge [-]\neg\varphi\big)$	4: $\big([+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge [-]\neg\varphi\big)$
5: $\big([+]\varphi \wedge [+]\neg\varphi\big) \wedge \big([-]\varphi \wedge \neg[-]\neg\varphi\big)$	6: $\big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big([-]\varphi \wedge [-]\neg\varphi\big)$
7: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge [-]\neg\varphi\big)$	8: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big([-]\varphi \wedge \neg[-]\neg\varphi\big)$
9: $\big([+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge [-]\neg\varphi\big)$	10: $\big([+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big([-]\varphi \wedge \neg[-]\neg\varphi\big)$
11: $\big([+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$	12: $\big([+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$
13: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$	14: $\big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big([-]\varphi \wedge \neg[-]\neg\varphi\big)$
15: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$	16: $\big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$

We explain these cases in terms of the mental attitudes they describe. Case 1 represents an agent's indecision about both $\neg\varphi$ and φ. Cases 2 and 3 correspond to the positive attitude towards $\neg\varphi$ and φ, respectively, whereas cases 4 and 5 give the corresponding negative attitudes. Case 6 corresponds to having no positive opinion whether φ holds, and case 11 to having no negative opinion. Case 7 corresponds to being undecided about $\neg\varphi$, case 10 about φ. Case 8 corresponds to positive about $\neg\varphi$, and negative about φ (strengthening of $\neg\varphi$), case 9 is just the opposite (strengthening of φ). Case 12 corresponds to having no negative attitude but only positive attitude towards φ, case 13 corresponds to having no negative attitude towards φ, and positive attitude towards $\neg\varphi$. Case 14 corresponds to having no positive attitude but only negative attitude towards φ, case 15 corresponds to no positive attitude towards φ, but negative attitude towards $\neg\varphi$. Case 16 corresponds to having no opinion whatsoever about φ.

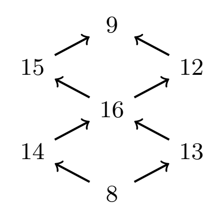
We can order these 16 attitudes according to their informational content, just like we did with the four attitudes that can be expressed with a single normal modality (cf. Section 2.1). The order goes, again, from the case in which the four relevant formulas are false (case 16) to the case in which all of them are true (case 1); we get the power set of a 4-element set ordered by inclusion:



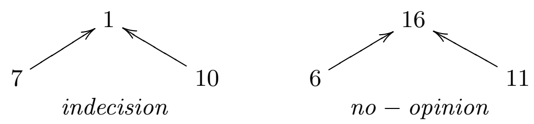
Of course, not all these cases are satisfiable in all classes of models:

– In $\mathbf{M}^+$-models, cases in $\{1, 4, 5, 11\}$ are *not* satisfiable.
– In $\mathbf{M}^-$-models, cases in $\{1, 2, 3, 6\}$ are *not* satisfiable.
– In $\mathbf{M}^\pm$-models, cases in $\{1, 2, 3, 4, 5, 6, 11\}$ are *not* satisfiable.
– In $\mathbf{M}^c$-models, *only* cases in $\{8, 9, 12, 13, 14, 15, 16\}$ are *satisfiable*.

In particular, the cases satisfiable in $\mathbf{M}^c$ can also be ordered according to the attitude towards a given formula, just like we did with the three $\Box$-consistent cases of the single normal modality (cf. Section 2.1). This time, the order goes from a completely negative attitude towards φ (case 8) to a completely positive attitude towards it (case 9), as the diagram on the right shows.



Moreover, considering the $\mathbf{M}$ models and the set of *uncertain* attitudes towards φ, viz. {1, 6, 7, 10, 11, 16}, we can have the following orderings of *indecision* and *no opinion*, in terms of increasing degrees of uncertainty. Note that these notions of uncertainty are independent of each other. The ordering for indecision and no opinion are as follows:



We should note here that for all practical purposes, the above representations are too long to comprehend. We consider their reduced versions corresponding to the $\mathbf{M}^\pm$ models:

1: $\big([+]\varphi \wedge [+]\neg\varphi\big) \wedge \big([-]\varphi \wedge [-]\neg\varphi\big)$ 2: $\big(\neg[+]\varphi \wedge [+]\neg\varphi\big)$ 3: $\big([+]\varphi \wedge \neg[+]\neg\varphi\big)$
4: $\big(\neg[-]\varphi \wedge [-]\neg\varphi\big)$ 5: $\big([-]\varphi \wedge \neg[-]\neg\varphi\big)$ 6: $\big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big)$
7: $[+]\neg\varphi \wedge [-]\neg\varphi$ 8: $[+]\neg\varphi \wedge [-]\varphi$ 9: $[+]\varphi \wedge [-]\neg\varphi$

$10\colon [+]\varphi \wedge [-]\varphi$ $\quad 11\colon \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$ $\quad 12\colon \big([+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$
$13\colon \big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$ $14\colon \big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big([-]\varphi \wedge \neg[-]\neg\varphi\big)$
$15\colon \big(\neg[+]\varphi \wedge [+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$ $16\colon \big(\neg[+]\varphi \wedge \neg[+]\neg\varphi\big) \wedge \big(\neg[-]\varphi \wedge \neg[-]\neg\varphi\big)$

We will have 15 different cases (case 1 being not satisfiable), preserving the corresponding attitudes they express. For our technical discussions we will consider all types of models and subsequently, 16 different cases.

As the readers can easily apprehend, though some of the cases are redundant, we have a plethora of notions of agent uncertainty like *indecision*, *no opinion*, and others. If we refer back to our example in the introduction, all the attitudes of Alice's mental state can now be expressed in the present framework. We consider the reduced versions. Alice is undecided about A and B: $([+]A \wedge [-]A) \wedge ([+]B \wedge [-]B)$, has strong positive opinion about C: $[+]C \wedge [-]\neg C$, negative opinion about D: $[-]D \wedge \neg[-]\neg D$, have not heard about E: $\big(\neg[+]E \wedge \neg[+]\neg E\big) \wedge \big(\neg[-]E \wedge \neg[-]\neg E\big)$.

3.3 The *K* system as a particular case

Classical modal logic assumes that having a positive attitude towards φ is the same as having a negative attitude towards $\neg\varphi$. A dual model in which the positive and the negative relations coincide satisfy this property.

Proposition 3.1. *Denote by $\boldsymbol{M}^{\Box}$ the class of dual models in which the positive and the negative relation are the same, that is, $R^+ = R^-$. in this class, the following formula is valid:*

$$[+]\varphi \leftrightarrow [-]\neg\varphi$$

Consider now the previous sixteen cases.

Proposition 3.2. *In the class $\boldsymbol{M}^{\Box}$, only cases 1, 8, 9 and 16 are satisfiable. In fact, by using the mentioned equivalence $[+]\varphi \leftrightarrow [-]\neg\varphi$, they become the four attitudes we can express with a single normal modality, that is,*

$$\begin{array}{llll} (1)\colon & [+]\varphi \wedge [+]\neg\varphi & (9)\colon & [+]\varphi \wedge \neg[+]\neg\varphi \\ (8)\colon & \neg[+]\varphi \wedge [+]\neg\varphi & (16)\colon & \neg[+]\varphi \wedge \neg[+]\neg\varphi \end{array}$$

Proposition 3.3. *The classical modal system K is a subsystem of the dual system obtained by using $\Box$ for $[+]$ and by asking for the positive and the negative relation to be the same.*

4 Profile of the *dual* framework

4.1 Decidability

We now show that, in general, we can always decide whether a given formula φ is **M**-valid or not. This is the case because our system, just like standard modal logic, has the *strong finite model property.*

Fact 1 *Let φ be a formula of $\mathcal{L}$. If φ is satisfiable in an $\boldsymbol{M}$-model, then it is satisfiable in a finite $\boldsymbol{M}$-model whose size is at most $2^{|\varphi|}$, where $|\varphi|$ is the size of the set of sub-formulas of φ closed under single negation.*

Moreover, the logic L has the strong finite model property with respect to a recursive set of **M**-models. So we have a method for deciding whether a formula φ is satisfiable or not. All we have to do is to verify if it is satisfiable in some pointed model of size up to $|\varphi|$ and, since there are only finitely many such models, we will eventually get an answer. If we find one such model, then φ is satisfiable; if not, then it follows from Fact 1 that φ is not satisfiable.

But in our system negation behaves classically, so a formula is valid if and only if its negation is not satisfiable. Hence, we can decide whether a formula φ is valid by deciding whether $\neg\varphi$ is satisfiable.

4.2 Complexity

We now provide complexity results for model checking and satisfiability problem in the *dual* framework. First, we state the problems formally.

Definition 4.1. *The* model checking *problem is, given a formula φ in $\mathcal{L}$ and a pointed dual model $(\langle W, R^+, R^-, V\rangle, w)$, decide whether φ is true at $(\mathcal{M}, w)$ or not. The* satisfiability *problem is, given a formula φ in $\mathcal{L}$ and a class of models $\boldsymbol{C}$, decide whether φ is satisfiable in a model of the class $\boldsymbol{C}$.*

The next fact provides us with an efficient algorithm for checking whether a formula is true in a pointed model. For the case of the satisfiability problem, we have already discussed an algorithm that decides whether a given formula is satisfiable in an **M**-model. The following gives a more efficient way to do it.

Fact 2 *In a* dual *framework, the complexity of model checking is P, while the complexity of satisfiability in the class of all dual models ($\boldsymbol{M}$) is PSPACE.*

4.3 Axiom systems

We can decide whether a given $\mathcal{L}$-formula is **M**-valid. Such formulas can also be syntactically characterized, as the following theorem shows.

Theorem 4.1. *The logic L, presented in the table below, is sound and complete for the language $\mathcal{L}$ with respect to models in $\boldsymbol{M}$.*

P	*All propositional tautologies*	**MP**	*If* $\vdash \varphi$ *and* $\vdash \varphi \to \psi$, *then* $\vdash \psi$
K+	$\vdash [+](\varphi \to \psi) \to ([+]\varphi \to [+]\psi)$	**K-**	$\vdash [-](\varphi \wedge \psi) \to ([-]\neg\varphi \to [-]\psi)$
Gen+	*If* $\vdash \varphi$, *then* $\vdash [+]\varphi$	**Gen-**	*If* $\vdash \neg\varphi$, *then* $\vdash [-]\varphi$

For the cases of $\mathbf{M}^+$ and $\mathbf{M}^-$, their validities are characterized by the logic L extended with the **D+** axiom $[+]\varphi \to \langle + \rangle\varphi$ (the logic L^+) and the **D-** axiom $[-]\varphi \to \langle - \rangle\varphi$ (the logic L^-), respectively. Naturally, validities for the class $\mathsf{L}^\pm$ are characterized by L extended with both **D+** and **D-** (the logic $\mathsf{L}^\pm$).

The case of the class $\mathbf{M}^c$ is different.

Theorem 4.2. *The logic* L^c*, extending* L *with the axiom* $[-]\varphi \rightarrow \neg[+]\varphi$*, is sound and complete for the language* $\mathcal{L}$ *with respect to* $\boldsymbol{M}^c$*-models.*

Finally, the case of validities in $\mathbf{M}^{\Box}$-models.

Theorem 4.3. *The logic* $\mathsf{L}^{\Box}$*, extending* L *with the axiom* $[+]\varphi \leftrightarrow [-]\neg\varphi$ *is sound and complete for the language* $\mathcal{L}$ *with respect to models in* $\boldsymbol{M}^{\Box}$*.*

5 Concrete interpretations

We now consider a particular interpretation of the $[+]$ and the $[-]$ modalities: *belief* and *disbelief*. Some intuitive ways to relate them are: *"disbelieving* φ*"* is a stronger notion than *"not believing in* φ*"*, whereas, *"believing in* $\neg\varphi$*"* should imply *"disbelieving* φ*"*. In fact, the reading of the different cases of dual expressions that we have provided in Section 3.3 is motivated by our understanding of $[+]$ as belief and $[-]$ as disbelief.

Consideration of disbelief as a separate epistemic category came to the fore in the latter part of last decade [20,3]. Consideration of changing or revising disbeliefs as a process analogous to belief revision was taken up by [4]. Belief-disbelief pairs, i.e. simultaneous consideration of belief and disbelief sets, were also taken up [2]. A more recent proposal can be found in [21], where 'disbelieving φ' is modeled as 'considering $\neg\varphi$ to be plausible'.

5.1 Belief-disbelief logic

We now propose the model and axiom system of the belief-disbelief logic ($\mathbf{L}_{KD45}$).

Definition 5.1. *We denote by* $\mathbf{M}_{KD45}$ *the class of models in* $\boldsymbol{M}$ *for which the positive and negative relations, now denoted by* R^{B} *and* R^{D}*, are serial, reflexive and Euclidean. Their respective universal modalities are given by* B *and* D *(with* $\widehat{\mathrm{B}}$ *and* $\widehat{\mathrm{D}}$ *denoting the corresponding existential ones).*

Theorem 5.1. *The logic* $\mathbf{L}_{KD45}$ *given by the axiom system of Theorem 4.1 plus the axioms below is sound and complete for* $\mathcal{L}$ *with respect to* $\mathbf{M}_{KD45}$*.*

D+	$\vdash \mathrm{B}\,\varphi \rightarrow \widehat{\mathrm{B}}\varphi$	**D-**	$\vdash \mathrm{D}\,\varphi \rightarrow \widehat{\mathrm{D}}\varphi$
4+	$\vdash \mathrm{B}\,\varphi \rightarrow \mathrm{B}\,\mathrm{B}\,\varphi$	**4-**	$\vdash \mathrm{D}\,\varphi \rightarrow \mathrm{D}\,\neg\,\mathrm{D}\,\varphi$
5+	$\vdash \neg\,\mathrm{B}\,\varphi \rightarrow \mathrm{B}\,\neg\,\mathrm{B}\,\varphi$	**5-**	$\vdash \neg\,\mathrm{D}\,\varphi \rightarrow \mathrm{D}\,\mathrm{D}\,\varphi$

What we have now is a minimal logic of belief and disbelief. To make things more interesting and useful we should have inter-relations between the belief and disbelief modalities. The table below lists interesting axioms and their corresponding characterizing-criteria in the $\mathbf{M}_{KD45}$ class of models.

C	$\vdash \mathrm{D}\,\varphi \rightarrow \neg\,\mathrm{B}\,\varphi$	**Mc**	$\forall w \in W,\ R^{\mathrm{B}}[w] \cap R^{\mathrm{D}}[w] \neq \emptyset$
BD	$\vdash \mathrm{B}\,\neg\varphi \rightarrow \mathrm{D}\,\varphi$	**Mbd**	$R^{\mathrm{D}} \subseteq R^{\mathrm{B}}$
DB	$\vdash \mathrm{D}\,\neg\varphi \rightarrow \mathrm{B}\,\varphi$	**Mdb**	$R^{\mathrm{B}} \subseteq R^{\mathrm{D}}$
Intro1	$\vdash \mathrm{D}\,\varphi \rightarrow \mathrm{B}\,\mathrm{D}\,\varphi$	**MI-1**	$wR^{\mathrm{B}}w' \wedge wR^{\mathrm{D}}w'' \Rightarrow w'R^{\mathrm{D}}w''$
Intro2	$\vdash \mathrm{B}\,\varphi \rightarrow \mathrm{D}\,\neg\,\mathrm{B}\,\varphi$	**MI-2**	$wR^{\mathrm{D}}w' \wedge wR^{\mathrm{B}}w'' \Rightarrow w'R^{\mathrm{B}}w''$

5.2 Preference

There is a very close relationship between an agent's beliefs and her preferences which has been extensively discussed in [22]. In fact, both objective and subjective preferences over objects are described, along with preference over propositions. The dual framework of belief-disbelief can also provide a way to describe subjective preferences over propositions, taking into account agents' uncertainties as well. For example, consider the following notion of preference.

$$Pref(\varphi, \psi) : (O_3\varphi \vee O_9\varphi) \wedge (O_5\psi \vee O_6\psi \vee O_{10}\psi \vee O_{16}\psi),$$

where $O_i\chi$ represents the i-th possibility of the 16 different attitudes described earlier. The $Pref$ relation defined above is *transitive* but *neither reflexive nor linear*. This is a very weak notion of preference in the sense that it cannot distinguish between longer preference orders, e.g., orders of length more than 3. But, if we go back to the decision-making scenario described in the introduction, we can still deduce from the known facts that $Pref(C, X)$, for $X = A, B, D, E$. We leave a more detailed discussion on stronger notions of preference as well as their technical study for future work.

6 A comparative discussion

As mentioned in the introduction, there already have been past works modeling these dual attitudes. In this section we provide a comparative discussion with two such proposals which are very close in spirit to ours.

6.1 A logic of acceptance and rejection

In [3], the authors present a nonmonotonic formalism AEL2 extending the framework of Moore's auto-epistemic logic [23] to deal with *uncertainty* of an agent. The underlying logical framework is the same as that of $\mathbf{L}_{KD45}$. In AEL2, accepted and rejected premises are separated to form a pair of sets of formulas (I_1, I_2). Then, the AEL2-extensions (T_1, T_2) are defined, where T_1 is expected to contain all the accepted formulas with respect to I_1 and T_2 to contain all the rejected formulas with respect to I_2.

Definition 6.1. (T_1, T_2) *is a stable* AEL2 *expansion of* (I_1, I_2) *if*

$$T_1 = Cn(I_1 \cup \{B\varphi : \varphi \in T_1\} \cup \{\neg B\varphi : \varphi \notin T_1\} \cup \{D\varphi : \varphi \in T_2\} \cup \{\neg D\varphi : \varphi \notin T_2\})$$
$$T_2 = Cn'(I_2 \cup \{\neg B\varphi : \varphi \in T_1\} \cup \{B\varphi : \varphi \notin T_1\} \cup \{\neg D\varphi : \varphi \in T_2\} \cup \{D\varphi : \varphi \notin T_2\}).$$

Here, Cn is the classical propositional consequence operator, and Cn' is the corresponding consequence operator for the propositional logic of contradictions.

Definition 6.2. (T_1, T_2) *is said to be a BD-dual extension of* (I_1, I_2) *if*

$$T_1 = \{\psi \mid I_1 \cup \{\neg B\varphi : \varphi \notin T_1\} \cup \{D\varphi : \varphi \in T_2\} \cup \{\neg D\varphi : \varphi \notin T_2\} \vdash_{\mathsf{L}_1} \psi\}$$
$$T_2 = \{\psi \mid I_2 \cup \{\neg B\varphi : \varphi \in T_1\} \cup \{B\varphi : \varphi \notin T_1\} \cup \{\neg D\varphi : \varphi \in T_2\} \vdash_{\mathsf{L}_2} \psi\}.$$

Here, L_1 is a fragment of $\mathbf{L}_{KD45}$ axiomatized by the propositional tautologies, MP, modal axioms **K+**, **4+**, and **5+**, and rule **Gen+** corresponding to the modal operator B, and L_2 is the logic axiomatized by the propositional contradictions, *Rej*, modal axioms **K-**, D_2, and D_3, and rule **Gen-** corresponding to the modal operator D, where D_2, D_3,and *Rej* are given below:

$$D_2 : D\varphi \to \neg DD\varphi \qquad \qquad Rej\colon \frac{\beta, \neg(\alpha \to \beta)}{\alpha}$$
$$D_3 : \neg D\varphi \to \neg D\neg D\varphi$$

It is a very well-known result that Moore's auto-epistemic logic corresponds to the non-monotonic modal logic weak $S5$, in particular the $K45$-logic (see [24] for a detailed discussion). Similarly, it can be proved that,

Proposition 6.1. *(T_1, T_2) is a consistent stable* AEL2 *expansion of (I_1, I_2) iff (T_1, T_2) is a BD-dual extension of (I_1, I_2).*

6.2 Some logics of belief and disbelief

In [2], the authors present several logics for dealing with beliefs and disbeliefs from a syntactic perspective, together with providing a neighbourhood-like semantics for them. Given a set of atomic propositions P, denote by L_B the set of classical propositional logic formulas built from P, and define $L_D := \{\overline{\phi} \mid \phi \in L_B\}$. The underlying language L is then given by $L_B \cup L_D$. Suppose $\Gamma \subseteq L$. The agent *believes* every ϕ with $\phi \in \Gamma$ (denoted by Γ_B), and *disbelieves* every ϕ with $\overline{\phi} \in \Gamma$ (denoted by Γ_D). Based on specific closure properties of Γ_B and Γ_D, the authors define four different logics, but all of them can be interpreted on **WBD** models, tuples of the form $\langle M, \mathcal{N}\rangle$ where M is a set of propositional valuations and $\mathcal{N} \subseteq \wp(V)$ is a set of sets of propositional valuations (the particular logics are characterized by properties of the semantic model). Then,

$\langle M, \mathcal{N}\rangle \models \phi$ iff ϕ is true under every valuation in M, and
$\langle M, \mathcal{N}\rangle \models \overline{\phi}$ iff $\neg\phi$ is true under every valuation of N *for some* $N \in \mathcal{N}$.

We now provide a semantic comparison between these logics and our framework. Consider a **WBD** model $\mathsf{M} = \langle M, \mathcal{N}\rangle$ in which $\mathcal{N}$ is finite, and denote by k its number of elements. We will build an extension of our dual models in which the domain consists of all the possible valuations for the given atomic propositions.

Definition 6.3. *Let P be a set of atomic propositions and let $\mathsf{M} = \langle M, \mathcal{N}\rangle$ be a **WBD** model based on them, with $\mathcal{N} = \{N_1, \ldots, N_k\}$ (i.e, $\mathcal{N}$ is finite). Denote by V the set of all propositional valuations over P, and denote by V_p the set of propositional valuations in V that make p true. The extended dual model $\mathcal{M}_{\mathsf{M}} = \langle W, R^{\mathrm{B}}, R^{\mathrm{B}}_s, R^{\mathrm{D}}, V\rangle$ has as domain the set of all valuations for P, that is, $W := \mathsf{V}$. Now select arbitrary $k+1$ worlds $\mathsf{w}, \mathsf{w}_1, \ldots, \mathsf{w}_k$ in W. Define $R^{\mathrm{B}}\mathsf{w}u$ iff $u \in M$. For each $i \in \{1, \ldots, k\}$, define $R^{\mathrm{D}}\mathsf{w}_i u$ iff $u \in N_i$. Define $R^{\mathrm{B}}_s\mathsf{w}\mathsf{w}_i$ for every $i \in \{1, \ldots, k\}$, and for every atomic proposition p, define $V(p) := \mathsf{V}_p$.*

For this special dual model, we use the modalities B, B_s and D for the relations R^{B}, R^{B}_s and R^{D}, respectively. Then, for every world $w \in W$,

$$
\begin{array}{lll}
(\mathcal{M}_{\mathsf{M}}, w) \models \mathrm{B}\,\varphi & \text{iff} & \text{for all } w' \text{ such that } R^{\mathrm{B}}ww',\ (\mathcal{M}_{\mathsf{M}}, w') \models \varphi \\
(\mathcal{M}_{\mathsf{M}}, w) \models \mathrm{B}_s\varphi & \text{iff} & \text{for all } w' \text{ such that } R^{\mathrm{B}}_s ww',\ (\mathcal{M}_{\mathsf{M}}, w') \models \varphi \\
(\mathcal{M}_{\mathsf{M}}, w) \models \mathrm{D}\,\varphi & \text{iff} & \text{for all } w' \text{ such that } R^{\mathrm{D}}ww',\ (\mathcal{M}_{\mathsf{M}}, w') \models \neg\varphi
\end{array}
$$

Proposition 6.2. *Let $\mathsf{M} = \langle M, \mathcal{N}\rangle$ be a **WBD** with $\mathcal{N}$ finite. For every propositional formula γ, we have*

$$\mathsf{M} \models \gamma \text{ iff } (\mathcal{M}_{\mathsf{M}}, \mathsf{w}) \models \mathrm{B}\,\gamma \qquad \mathsf{M} \models \overline{\gamma} \text{ iff } (\mathcal{M}_{\mathsf{M}}, \mathsf{w}) \models \widehat{\mathrm{B}_s}\,\mathrm{D}\,\gamma$$

In other words, our formula $\mathrm{B}\,\gamma$ expresses the notion "the agent believes γ" of [2], and our formula $\widehat{\mathrm{B}_s}\,\mathrm{D}\,\gamma$ expresses their "the agent disbelieves γ".

In [25], the authors suggest combining universal and existential notions to describe *knowledge*. The above proposition puts us on similar track, but we leave the detailed study for future work.

7 Dynamics

The system proposed in Section 3 allows us to represent positive and negative attitudes by means of two modalities that allow us to build formulas of the form $[+]\varphi$ and $[-]\varphi$. While the first one is true at a world w iff φ is true in all the worlds R^+– reachable from w, the second is true at a world w iff φ is *false* in all the worlds R^-– reachable from w. We have also shown that, when the two relations R^+ and R^- are the same, we get the validity $[-]\varphi \leftrightarrow [+]\neg\varphi$ indicating that the agent has a negative attitude towards a formula iff she has a positive attitude towards its negation. This actually says that, when $R^+ = R^-$, the negative attitude collapses into classical negation, and therefore we get the classical K-system.

But from a more dynamic perspective, the case in which $R^+ = R^-$, that is, the K case can be thought of as not a particular case of the static system, but a possible result of some dynamic extension. In other words, the 'ideal' system K in which negative attitudes coincide with classical negation, can be seen not as the state of an ideal static agent, but as the possible final state of a non-ideal but dynamic one who can perform actions that make the two relations the same. This section looks briefly at possible results of such actions.

There are various ways to generate a new relation from two others, and in this case they represent the different policies through which the agent 'merges' her positive and negative attitudes. For example, she can be drastic in two different ways: give up her negative attitude ($R := R^+$) or give up the positive one ($R := R^-$). More reasonable are the policies that actually combine the two relations, like $R := R^+ \cup R^-$.

Definition 7.1 (Merging policies). *Let $\mathcal{M} = \langle W, R^+, R^-, V\rangle$ be a dual model. The relation R of a dual model $\langle W, R, V\rangle$ that results from the agent's merging of her positive and negative attitudes can be defined in several forms.*

– $R := R^+$ *(the* drastic positive *policy; new model denoted by* $\mathcal{M}_+$*).*
– $R := R^-$ *(the* drastic negative *policy; new model denoted by* $\mathcal{M}_-$*).*
– $R := R^+ \cup R^-$ *(the* liberal combining *policy; new model denoted by* $\mathcal{M}_\cup$*).*
– $R := \alpha(R^+, R^-)$*, where* $\alpha(R^+, R^-)$ *is a PDL-expression [26] based on* R^+ *and* R^- *(the* PDL *policy; new model denoted by* $\mathcal{M}_\alpha$*).*
– $R := R^+ \cap R^-$ *(the* skeptic combining *policy; new model denoted by* $\mathcal{M}_\cap$*).*
– $R := R^+ \setminus R^-$ *(new model denoted by* $\mathcal{M}_\pm$*).*
– $R := R^- \setminus R^+$ *(new model denoted by* $\mathcal{M}_\mp$*).*

For each policy $\circ$, we define a modality $[m_\circ]$ for building formulas of the form $\langle m_\circ \rangle\varphi$, read as "there is a way of merging attitudes with policy $\circ$ after which φ is the case". Their semantic interpretation is given by:

$$(\mathcal{M}, w) \models \langle m_\circ \rangle\varphi \quad \text{iff} \quad (\mathcal{M}_\circ, w) \models \varphi$$

Now, for an axiom system, we can provide reduction axioms for each policy. In each case, the relevant ones are those describing the way the new relations are created. Though they are the same, we will stick with $+$ and $-$ for representing the positive and negative relations after the operation.

The *drastic positive* policy:	$\langle m_+ \rangle\langle + \rangle\varphi \leftrightarrow \langle + \rangle\langle m_+ \rangle\varphi$
	$\langle m_+ \rangle\langle - \rangle\varphi \leftrightarrow \langle + \rangle\langle m_+ \rangle\neg\varphi$
The *drastic negative* policy:	$\langle m_- \rangle\langle + \rangle\varphi \leftrightarrow \langle - \rangle\langle m_- \rangle\neg\varphi$
	$\langle m_- \rangle\langle - \rangle\varphi \leftrightarrow \langle - \rangle\langle m_- \rangle\varphi$
The *liberal combining* policy:	$\langle m_\cup \rangle\langle + \rangle\varphi \leftrightarrow \langle + \rangle\langle m_\cup \rangle\varphi \vee \langle - \rangle\langle m_\cup \rangle\neg\varphi$
	$\langle m_\cup \rangle\langle - \rangle\varphi \leftrightarrow \langle + \rangle\langle m_\cup \rangle\neg\varphi \vee \langle - \rangle\langle m_\cup \rangle\varphi$

For $*$-free *PDL* policies, reduction axioms for each particular $\alpha(R^+, R^-)$ can be obtained by following the technique introduced in [27]. To get sound and complete reduction axioms for the policies involving $\cap$ and $\setminus$, we may need to extend the language with nominals.

But we do not only need to look at actions that create a single relation in one shot. We can also look at procedures in which the single relation is a long-term result of small operations that merge the two of them in a step-wise form. We leave the detailed study on these dynamical aspects for future work.

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