Revisiting games in dynamic-epistemic logic

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Abstract: We revisit the discussion on reasoning *about* games in dynamic-epistemic logic and present a language for describing reasoning *in* possibly infinite games from the perspective of the players. We argue that even though a plethora of sophisticated logics of strategic reasoning in games are available, it is still worthwhile to consider the game structures themselves from the perspective of logic. In the process, we provide complete axiom systems for these games satisfying characteristic properties from the game-theoretic literature. Decidability of the satisfiability problem is also taken up to consider the existence of games following certain rules that can be expressed in the logical language.

1 Introduction

Dynamic games (Osborne and Rubinstein, 1994) provide us with faithful models of interactive multi-agent systems (Wooldridge, 2009), and as such, investigating such game structures falls under the purview of AI, computer science, logic and other related areas, in addition to their in-depth investigations in game theory. These games have rich transition structures and form a fertile research field from a logician's view point, especially for modal logicians. Reasoning about n-player extensive-form games with perfect and imperfect information form a major sub-area which provides us with natural models for dynamic-epistemic languages. In particular, a combination of propositional dynamic logic (Harel et al., 2000) and epistemic logic (Fagin et al., 1995) provides a suitable framework to reason *about* these games from the players' local action perspectives (e.g. see (van Benthem, 2001)). These frameworks are also suitable to model reasoning *in* games from the viewpoint of the players playing the game.

Current literature abounds with various logics of games and strategies. Their reasoning prowess proceeds on several levels: players' long-term powers in games in general (Pauly and Parikh, 2003), players' powers in parallel games (van Benthem et al., 2008; Ghosh et al., 2010), and coalitional abilities of players in short-term games (Pauly, 2001) as well as in games of unbounded duration (Alur et al., 2002). With regard to explicit reasoning about strategies, bringing strategic reasoning to the fore, numerous logics have also been developed (e.g., see (Chatterjee et al., 2007; Walther et al., 2007; Ramanujam and Simon, 2008; Brihaye et al., 2009), to name a few). More recently, many of these logics have been extended with different *knowledge* operators representing individual, general, distributed, and common knowledge to describe information available to the players (e.g., see (Ågotnes and Alechina, 2012; Belardinelli, 2014; Berthon et al., 2017)). With respect to extensive-form games of perfect and imperfect information, reasoning about long term players' powers, short term action abilities of players, structural as well higher level game equivalences have been taken up by van Benthem (2001, 2002). The current work proceeds in this direction with the aim of bringing in more game-theoretic intricacies in the logical development.

The main aspect that we look into is that of information available to players. If we consider any game-theoretic literature defining extensive-form games (e.g. (Osborne and Rubinstein, 1994)), we come across the following concept: For any player i, an equivalence relation, $\sim_i$, say, is defined among the nodes of player i to model the concept of indistinguishability for player i. For two game nodes t and t', say, of player i, the interpretation of $t \sim_i t'$ is that they are indistinguishable to player i. In other words, as per the knowledge of player i, she could be making a decision either at node t or at node t'. Any such equivalence class is called an *information set* for player i. It is also assumed that if any action of player i is available at one node of an information set then it is also available at every node in that information set. Such indistinguishability of nodes for players is generally modelled in logic by an *S5 knowledge* operator (Fagin et al., 1995) for each player i, which is defined for all nodes of a game, not just the player i nodes. So,

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a natural question is: How can one construct a reasonable extension of the definition of the equivalence relation $\sim_i$ for player i from the set of nodes of player i to the set of all nodes of the game so that modelling by a knowledge operator can be facilitated? This has generally been taken care of by assuming some arbitrary equivalences over the set of all nodes of a game (van Benthem, 2001) or some arbitrary property of the knowledge operator (Bonanno, 2003, 2004a). A notable exception is the work done by Bonanno and Battigalli (1997), which acknowledges the fact that the language of extensive-form games as available in the game-theoretic literature is not rich enough to deal with many natural and meaningful statements that one can make about such games. A natural extension of these information sets satisfying certain properties is defined over the set of all nodes in Von Neumann games (Kuhn, 1953) and the existence of such extensions is proved corresponding to these games. A converse result is proved by Bonanno (2004b), who shows that the games on which these extensions can be defined are exactly the Von Neumann games.

The main goal of this work is to bring a general notion of such extensions under the purview of logic and to investigate various related properties of these games which have been studied over the years. We first provide a sound and complete axiomatization of the general game structures where these extensions are defined in a faithful manner corresponding to the information sets of the players. We also discuss the expressive power of our language. Then we provide a sound and complete axiom system for games with perfect recall, more specifically, memory recall and action recall. We note here that Bonanno (2003, 2004a) has provided syntactic characterizations of these properties and Witzel (2011a,b) has provided a detailed study of perfect recall semantics in the framework of epistemic temporal logics (Fagin et al., 1995; Parikh and Ramanujam, 2003). Our claim is that the axiomatizations that we provide here, distinctive for dynamic games of complete information, bring the logical systems closer to their game-theoretic counterparts, and thus pave the way for further studies on strategic reasoning in these dynamic games focusing on the underlying local structures. Such structures often lead to composition of partial strategies for players while playing the game, especially when the end is not known and future plays cannot be predicted. This work provides certain possibilities for studying these partial strategies in a logic framework, depending on the local structure of the games. We also show decidability of the satisfiability problem, which provides us with possibilities of checking on the existence of different rule structures for designing games of this nature.

Before proceeding any further, we would like to mention that dynamic games of complete information, which are the main focus of study in this work, have already been studied extensively in the framework of epistemic temporal logics (Fagin et al., 1995) and that of dynamic-epistemic logics (van Benthem, 2001, 2002; van Benthem et al., 2011). In fact, many of the results that we prove here have already been discussed and proved in the respective frameworks. The novelty of this paper lies in two points. The first one is that the question about reasonable extensions of information sets that we ask above has not yet been considered in these frameworks. The second point is that the language we use to describe these games and the corresponding axiom systems are tailor-made for such structures, facilitating the study of local strategic reasoning in these games.

In the remaining part of the paper, Section 2 provides the preliminary definitions of the game structures that we use in our study. Section 3 describes the logical syntax, discusses the expressivity of the language, provide a sound and complete axiomatization for the general extensive-form game structures, and show decidability of the satisfiability problem. Section 4 deals with extensive-form games with perfect recall and provide sound and complete axiomatizations for these structures. We round up our work in Section 5 with some discussion about future work.

2 Preliminaries

We start with defining our imperfect information game models based on game trees, where at most one player is allowed to move at a game position – these game positions are represented by nodes of the tree and the moves are represented by the labelled edges. Let N denote the finite set of players; we use i to range over this set. Let Σ be a finite set of action symbols representing moves of players; we let a, b range over Σ.

2.1 Game trees

Let $\mathbb{T}(\Sigma) = (S, \Rightarrow, s_0)$ be a Σ-labelled tree (edge-labelled tree) rooted at s_0 on the set of vertices S and let $\Rightarrow : (S \times \Sigma) \to S$ be a *partial* function specifying the edges of the tree. Thus for any $a \in \Sigma$, a labelled edge $\overset{a}{\Rightarrow}$ is a partial function from S to S. The tree $\mathbb{T}(\Sigma)$ is said to be finite if S is a finite set. For a node $s \in S$, let $\overrightarrow{s} = \{s' \in S \mid s \overset{a}{\Rightarrow} s'$ for some $a \in \Sigma\}$ and let *moves*$(s) = \{a \in \Sigma \mid \exists s' \in S$ with $s \overset{a}{\Rightarrow} s'\}$. An action "$a$" is said to be *enabled* at a node s if $a \in$ *moves*(s). A node s is called a *leaf* node (or terminal node) if $\overrightarrow{s} = \emptyset$. Let *frontier* denote the set of all leaf nodes. For the rest of the paper we fix Σ to be the finite set $\Sigma = \{a_1, \ldots, a_m\}$, we use $\mathbb{T}$ to denote the tree $\mathbb{T}(\Sigma)$.

An *extensive-form game tree* is a tuple $T = (S, \Rightarrow, s_0, \widehat{\lambda})$ where $\mathbb{T} = (S, \Rightarrow, s_0)$ is a tree. The set S denotes the set of game positions with s_0 being the initial game position. The edge function $\Rightarrow$ specifies the moves enabled at a game position and the turn function $\widehat{\lambda} : S \setminus$ *frontier* $\to N$ associates each non-leaf game position with a player. An extensive form game tree T is said to be finite if the underlying tree structure $(S, \Rightarrow, s_0)$ is finite. For $i \in N$, let $S^i = \{s \mid \widehat{\lambda}(s) = i\}$.

A *play* in the game T starts by placing a token on s_0 and proceeds as follows: at any stage if the token is at a position s and $\widehat{\lambda}(s) = i$, then player i picks an action which is enabled for her at s, and the token is moved to s' where $s \overset{a}{\Rightarrow} s'$. Formally, a play in T is a finite path $\rho : s_0 a_1 s_1 \cdots a_k s_k$ or an infinite path $\rho : s_0 a_1 s_1 \cdots$ in the underlying tree $\mathbb{T}$ such that for all $j > 0$, $s_{j-1} \overset{a_j}{\Rightarrow} s_j$.

2.2 Imperfect information

An *extensive-form game with imperfect information* is given by the tuple $T_{\mathcal{I}} = (T, \{\sim_i^T\}_{i \in N})$, where T is an extensive form game tree as defined above, and for each $i \in N$, $\sim_i^T$ is an equivalence relation over S^i. For each $s \in S^i$, let $[s]_{\sim_i^T} = \{s' : s \sim_i^T s'\}$, the equivalence class of s. Each such equivalence class is called an *information set* for player i. Thus, in game theory (Osborne and Rubinstein, 1994), information sets for a player i are defined over S^i, the set of all player i-nodes. To express such game structures in modal logics, we need to define these information sets for each player over the set of all nodes, S.

While giving syntactic characterizations of game-theoretic properties of imperfect information games, a standard assumption is that 'player i knows everything at the nodes where they are not making any move' (e.g., see (Bonanno, 2003, 2004a)), which is somewhat non-intuitive. Going with the spirit of the work done in (Bonanno and Battigalli, 1997; Bonanno, 2004b), we define an *extensive-form game with extended information* to be the tuple $\hat{T_{\mathcal{I}}} = (T_{\mathcal{I}}, \{\approx_i^T\}_{i \in N})$, where $T_{\mathcal{I}}$ is defined as above, and $\approx_i^T$ is an equivalence relation over S in $T_{\mathcal{I}}$ extending $\sim_i^T$ and satisfying the following condition: For all $s \in S^i$, $[s]_{\approx_i^T} = [s]_{\sim_i^T}$, where $[s]_{\approx_i^T}$ denotes the equivalence class of s under the relation $\approx_i^T$. These extensive-form games with extended information are the game structures in the models that we use in the subsequent sections.

2.3 Von Neumann games

We now define Von Neumann games (Kuhn, 1953, 1997) which play a significant role in the later part of the technical discussion in the current paper. They are very similar to the synchronous structures studied in the *interpreted systems* literature (e.g. see (Fagin et al., 1995)).

Let $T_{\mathcal{I}} = (T, \{\sim_i^T\}_{i \in N})$ denote an extensive-form game with imperfect information, where $T = (S, \Rightarrow, s_0, \widehat{\lambda})$. For every node $s \in S$, let $l(s)$ denote the number of predecessors of s, that is, the length of the path from the root s_0 to s. An extensive-form game is said to be Von Neumann if, whenever s and s' are decision nodes of player i that belong to the same information set of player i, the number of predecessors of s is equal to the number of predecessors of s': For all $i \in N$, for all $s, s' \in S$, if $s \sim_i^T s'$ then $l(s) = l(s')$.

3 A logic for extensive-form games

We now present a language to describe the extensive-form game trees with extended information defined above, followed by a complete axiom system for these game structures.

3.1 Syntax and semantics

Given a countable set of atomic propositions $\mathbb{P}$, a finite set of action symbols Σ, a finite set of players N, formulas α of the language $\mathcal{L}$ are defined as follows:

$$\alpha := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle a\rangle\alpha \mid \langle\overline{a}\rangle\alpha \mid \diamondminus\alpha \mid \Diamond\alpha \mid K_i\alpha,$$

where $p \in \mathbb{P}$, $a \in \Sigma$, $i \in N$, $\overline{a}$ denotes the converse of the action a, K_i denotes the knowledge modality of player i, and $\diamondminus$ and $\Diamond$ denote the past and future modalities, respectively.

Our main aim for considering this syntax is to have a parsimonious language to describe players' reasoning while playing the games. The action and knowledge modalities are the minimal requirements for such purposes. We have past modalities to talk about the history of the game at the current node, an essential ingredient of an extensive-form game. The only operator that does not really fit our mentioned aim of parsimony in the syntax is the future modality, but we need that for a technical purpose, to ensure completeness of the proposed logic.

The derived connectives $\wedge$ and $\supset$ are defined as usual. Let $\boxminus\alpha = \neg\diamondminus\neg\alpha$, $\Box\alpha = \neg\Diamond\neg\alpha$, $\langle N\rangle\alpha = \bigvee_{a\in\Sigma}\langle a\rangle\alpha$, $[N]\alpha = \neg\langle N\rangle\neg\alpha$, $\langle P\rangle\alpha = \bigvee\limits_{a\in\Sigma}\langle\overline{a}\rangle\alpha$, and $[P] = \neg\langle P\rangle\neg\alpha$. Let $[a]\alpha = \neg\langle a\rangle\neg\alpha$, $[\overline{a}]\alpha = \neg\langle\overline{a}\rangle\neg\alpha$ and $L_i\alpha = \neg K_i\neg\alpha$.

We use some special atoms: $\mathbf{turn}_i$ for each $i \in N$, define **root** $:= \neg\langle P\rangle\top$, **leaf** $:= \neg\langle N\rangle\top$.

Models of the logic are of the form $M = (T, \{\sim_i^T\}_{i\in N}, \{\approx_i^T\}_{i\in N}, V)$ (extensive-form game with extended information with valuations), where $T = (S, \Rightarrow, s_0, \widehat{\lambda})$ is an extensive form game tree, for each $i \in N$ and $\sim_i^T$ is an equivalence relation on $S^i \subseteq S$ (the set of all player i nodes), that is, $(T, \{\sim_i^T\}_{i\in N})$ gives an extensive-form game tree with imperfect information. For each $i \in N$, $\approx_i^T$ is an equivalence relation extending $\sim_i^T$ to the set of all nodes satisfying the condition mentioned in Section 2.2, and we thus have an extensive-form game with extended information. Here, $V : S \to 2^P$ is a valuation function, satisfying the following condition corresponding to the decision nodes of the game:

- For all $s \in S$ and $i \in N$, $\mathbf{turn}_i \in V(s)$ iff $\widehat{\lambda}(s) = i$.

The truth of a formula $\alpha \in \mathcal{L}$ in a model M and position s (denoted $M, s \models \alpha$) is defined by induction on the structure of α, as usual. For $s, s' \in S$, let $\rho_s^{s'}$ denote the path from s to s', $s\overset{a_0}{\Rightarrow}s_1 \cdots \overset{a_{m-1}}{\Rightarrow} s_m = s'$. In particular, for any $s \in S$, there exists a path from the root s_0 to s, which we denote by $\rho_{s_0}^s$. Another way to describe these paths is to consider the reflexive, transitive closure $\Rightarrow^*$ of the move relation $\Rightarrow$.

- $M, s \models p$ iff $p \in V(s)$.
- $M, s \models \neg\alpha$ iff $M, s \not\models \alpha$.
- $M, s \models \alpha_1 \vee \alpha_2$ iff $M, s \models \alpha_1$ or $M, s \models \alpha_2$.
- $M, s \models \langle a\rangle\alpha$ iff there exists s' such that $s\overset{a}{\Rightarrow}s'$ and $M, s' \models \alpha$.
- $M, s \models \langle\overline{a}\rangle\alpha$ iff $m > 0$, $a = a_{m-1}$ in $\rho_{s_0}^s$ and $M, s_{m-1} \models \alpha$.
- $M, s \models \diamondminus\alpha$ iff there exists s_j, $0 \leq j \leq m$, in $\rho_{s_0}^s$ such that $M, s_j \models \alpha$.
- $M, s \models \Diamond\alpha$ iff there exists s' such that $s\Rightarrow^* s'$ and $M, s' \models \alpha$.
- $M, s \models K_i\alpha$ iff for all s' such that $s \approx_i^T s'$, $M, s' \models \alpha$.

A formula α is said to be satisfiable if there exists a model M and a state s in the model such that $M, s \models \alpha$. A formula α is said to be true in a model M, denoted by $M \models \alpha$, if for all states s in M, $M, s \models \alpha$. A formula α is said to be valid if for all models M, $M \models \alpha$.

3.2 Reasoning *in* games

We have defined the syntax and semantics of our logic. The next questions that come up are: What kind of reasoning of players can we express with the language under consideration? To what extent can we express strategizing of players in extensive form games? Can we integrate knowledge of players and their strategies in a meaningful way? We answer these questions by providing suitable exemplifications regarding what we can express with the syntax described above.

Let us first give a few specific examples on what the knowledge modality brings in terms of describing information available to the players.

- Any player knows whenever she is playing: $\mathbf{turn}_i \supset K_i \mathbf{turn}_i$
- Every player knows whoever is playing: $\bigwedge_i (\mathbf{turn}_i \supset (\bigwedge_j K_j \mathbf{turn}_i))$

In terms of knowledge and history of the game, players can get some access to the opponents' earlier moves.

- If player i knows that her current node is reached by an a move then she knows that player j must have played in the past and move b was enabled: $(\mathbf{turn}_i \wedge K_i \langle \overline{a} \rangle \top) \supset (K_i \diamondminus (\mathbf{turn}_j \wedge \langle b \rangle \top))$

In terms of knowledge and future of the game, players can ensure certain objectives while playing the game.

- If player i knows p at the current node then whenever she is playing she can move to a node where action a is unavailable: $K_i p \supset \Box(\mathbf{turn}_i \supset \langle N \rangle [a] \bot)$

Strategic response can also be modelled in this logic.

- If player i knows that player j has been playing b in the history of the game, then player i can play a at the current node: $(\mathbf{turn}_i \wedge K_i \boxminus (\mathbf{turn}_j \supset \langle b \rangle \top)) \supset \langle a \rangle \top$

Thus, the logical framework is rich enough to deal with strategic reasoning of players at the local level incorporating information available to the players.We now provide a complete axiomatization of the proposed logic. For the sake of simplicity, we assume that all actions are available to all the players at their decision nodes. In terms of applications in designing games, these axioms would provide a safety-check regarding the properties that any game structure should satisfy.

3.3 Completeness and decidability

The axiom system that we propose below is different from the one provided in (van Benthem, 2001) which also dealt with games in dynamic-epistemic logic – an obvious reason is the underlying language, but more importantly, the main difference lies in the fact that we consider trees with extended information sets, instead of arbitrary structures with move relations (graph edges) and knowledge relations (equivalences). Our main focus here is to construct a system where we can describe game-theoretic intricacies in a more detailed manner, dwelling upon the facts that the underlying structure is that of a tree and that each extended information set is a specific extension (cf. Section 2.2) of the usual information sets.

Axioms

(A0) (a) All substitutional instances of the tautologies of classical propositional logic.
(b) $\mathbf{leaf} \equiv \neg(\bigvee_i \mathbf{turn}_i)$.
(c) $\mathbf{turn}_i \equiv \neg(\mathbf{leaf} \vee (\bigvee_{j \neq i} \mathbf{turn}_j))$.

(A1) (a) $[a](\alpha_1 \supset \alpha_2) \supset ([a]\alpha_1 \supset [a]\alpha_2)$.
(b) $[\overline{a}](\alpha_1 \supset \alpha_2) \supset ([\overline{a}]\alpha_1 \supset [\overline{a}]\alpha_2)$.
(c) $K_i(\alpha_1 \supset \alpha_2) \supset (K_i\alpha_1 \supset K_i\alpha_2)$.

(A2) (a) $\langle a\rangle\alpha \supset [a]\alpha$.
(b) $\langle\overline{a}\rangle\alpha \supset [\overline{a}]\alpha$.
(c) $\langle\overline{a}\rangle\top \supset \neg\langle\overline{b}\rangle\top$ for all $b \neq a$.

(A3) (a) $\alpha \supset [a]\langle\overline{a}\rangle\alpha$.
(b) $\alpha \supset [\overline{a}]\langle a\rangle\alpha$.

(A4) (a) $\diamondminus$ **root**.
(b) $\boxminus\alpha \equiv (\alpha \wedge [P]\boxminus\alpha)$.
(c) $\Box\alpha \equiv (\alpha \wedge [N]\Box\alpha)$.

(A5) (a) $K_i\alpha \supset \alpha$.
(b) $K_i\alpha \supset K_iK_i\alpha$.
(c) $\neg K_i\alpha \supset K_i\neg K_i\alpha$.

(A6) (a) $\mathbf{turn}_i \supset K_i\mathbf{turn}_i$.
(b) $L_i\alpha \supset \diamondminus\Diamond\alpha$

Inference rules

$(MP)\ \dfrac{\alpha,\ \ \alpha \supset \beta}{\beta}\qquad (GF)\ \dfrac{\alpha}{[a]\alpha}\qquad (GP)\ \dfrac{\alpha}{[\overline{a}]\alpha}\qquad (GK)\ \dfrac{\alpha}{K_i\alpha}$

$(Past)\ \dfrac{\alpha \supset [P]\alpha}{\alpha \supset \boxminus\alpha}\qquad (Future)\ \dfrac{\alpha \supset [N]\alpha}{\alpha \supset \Box\alpha}$

The axioms (A0) and (A1) need no explanation. Axioms (A2) take care of the determinacy of the actions. Axioms (A3) provide the converse properties. Axiom (A4)(a) takes care of the root node. Axioms (A4)(b) and (c) and rules (*Past*) and (*Future*) take care of the past and future formulas and allow induction. Axioms (A5) ensure equivalence of the knowledge relation. The special axioms (A6) are needed for our particular game structure. The system is a modified version of the one provided in (Ramanujam and Simon, 2008) which focuses on structures in strategies in perfect information games. A detailed survey of such frameworks for perfect information games can be found in (Ghosh and Ramanujam, 2012). We incorporated necessary changes to deal with reasoning in general extensive-form games.

Theorem 1 *The axioms (A0) - (A6) and the rules (MP), (GF), (GP), (GK), (Past), (Future) provide a sound and complete axiomatization of $\mathcal{L}$.*

Remark: It follows that the axioms and rules given above provide a sound and complete axiomatization of general dynamic games of complete information. This is because of the fact that any extensive-form game with extended information is also a dynamic game of complete information.

By the satisfiability problem for $\mathcal{L}$ we mean to consider the following question: Given a formula α in the language of $\mathcal{L}$, is α satisfiable? The following theorem shows that for $\mathcal{L}$, this question can be solved finitarily. The proof equips us with an algorithm to decide whether a formula depicting a certain property or a rule can actually be implemented in a game.

Theorem 2 *The satisfiability problem for $\mathcal{L}$ is decidable in non-deterministic double exponential time.*

We finish this section acknowledging the fact that neither the completeness nor the decidability result comes as a surprise, given the corresponding results for epistemic temporal logics modelling interpreted systems, which can model these game trees along with many other systems (cf. Fagin et al. (1995)). The novelty lies in the language we use to model these games, and hence in the axiom system, which we believe to be closer in spirit to the games they model, and in the proof techniques, influenced by the language under consideration.

4 Perfect recall: Knowledge memory and action recall

Perfect recall is a natural assumption of game theorists when it comes to the study of extensive-form games in general. This concept was introduced by Kuhn (1953). His interpretation of the concept is as follows: “each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves”. The following definition of perfect recall (*PR*) was provided by Selten (1975), which says that if a node y of player i is reached by an a move of player i from another node t of player i, then any node belonging to the information set of y can be reached by some node belonging to the information set of t following an a move of player i:

(*PR*): For every player $i \in N$, and for all nodes $t, y, y' \in S^i$ and $x \in S$ and for every action a, if $t \overset{a}{\Rightarrow} x$, $x \Rightarrow^* y$ and $y \sim_i^T y'$ then there exist nodes $t' \in S^i$, and $x' \in S$ such that $t \sim_i^T t'$, $t' \overset{a}{\Rightarrow} x'$ and $x' \Rightarrow^* y'$.

Various syntactic representations of perfect recall are present in the literature (e.g., see (van Benthem, 2001; Bonanno, 2003, 2004a)). What we provide here is a sound and complete axiom system for extensive-form games with extended information together with the property of perfect recall. Even though we consider games with extended information the notion of pefect recall for a player i is defined in terms of player i nodes, and not in terms of all the nodes (cf. Section 4.1). The main reason is that we wanted to have axioms corresponding to the property of perfect recall as it is defined in game theory.

Taking Kuhn’s interpretation of the concept of perfect recall into account, as Bonanno (2004a) points out, one can consider two independent components, that of *knowledge memory* (*KM*) and *action recall* (*AR*), which constitute the concept. For more details on these two concepts, see (Bonanno, 2004a) – we provide the relevant definitions here.

(*KM*): For every player $i \in N$ and for all nodes $s, t, t' \in S^i$, if $s \Rightarrow^* t$ and $t \sim_i^T t'$, then there exists a node $s' \in S^i$ such that $s \sim_i^T s'$, and $s' \Rightarrow^* t'$.

(*AR*): For every player $i \in N$, for all actions $a \in \Sigma$ and for all nodes $s, t, t' \in S^i$ and $x \in S$, if $s \overset{a}{\Rightarrow} x$, $x \Rightarrow^* t$ and $t \sim_i^T t'$, then there exist nodes $s' \in S^i$ and $x' \in S$ such that $s' \overset{a}{\Rightarrow} x'$, and $x' \Rightarrow^* t'$.[1]

Note the subtle differences between (*PR*) and the concepts defined by (*KM*) and (*AR*). As the names suggest, (*KM*) deals with the information sets in the past and (*AR*) deals with the actions in the past. The syntactic characterizations given in (Bonanno, 2004a) provide us with the ideas of the following axioms, which we show to be sound and complete with respect to the corresponding game structures under consideration :

(A_{KM}): $\Diamond(\mathbf{turn}_i \wedge L_i\alpha) \supset L_i\Diamond\alpha$.

(A_{AR}): $\langle\overline{a}\rangle\mathbf{turn}_i \supset \Box K_i(\langle\overline{a}\rangle\mathbf{turn}_i \vee \Diamond\!\!\!\!-\langle\overline{a}\rangle\mathbf{turn}_i)$.

Proposition 3 *The complete axiom system for $\mathcal{L}$ together with (A_{KM}) provide a complete axiom system for extensive-form games with extended information satisfying knowledge memory.*

Proposition 4 *The complete axiom system for $\mathcal{L}$ together with (A_{AR}) provide a complete axiom system for extensive form games with extended information satisfying action recall.*

Corollary 5 *The complete axiom system for $\mathcal{L}$ together with (A_{KM}) and (A_{AR}) provide a complete axiom system for extensive-form games with extended information satisfying perfect recall.*

Remark: It follows that the axioms and rules given above provide a sound and complete axiomatization of general dynamic games of complete information with perfect recall. This is because of the fact that any extensive-form game with extended information satisfying perfect recall is also a dynamic game of complete information satisfying perfect recall.

We note that the concept of knowledge memory (termed as *perfect recall* in (Fagin et al., 1995)) has also been studied extensively in the epistemic temporal logic literature for both synchronous as well as general systems providing axioms

[1] This definition is a weaker version of the one provided in Bonanno (2004a) in the sense that it does not require a player to remember how often she has performed a particular action, but in presence of the condition (*KM*), the proof of Corollary 5 goes through.

for the same (van der Meyden, 1994; Fagin et al., 1995). Here, we combine knowledge memory with action recall, to get the notion of perfect recall as it is known in the game-theoretic literature.

Even though we are considering games with extended information, the notion of perfect recall for a player i is defined with respect to the usual information sets in the game, that is, the nodes that are considered to define knowledge memory and action recall for player i are the nodes for player i only. But what about the corresponding concepts when all nodes of a game are taken under consideration? We take up this issue in the following section.

4.1 Von Neumann games: Broadening focus vis-à-vis restricting structures

We now consider extending the notion of perfect recall in games with extended information to all nodes of the game. This issue comes up in (Bonanno and Battigalli, 1997), where it is proved that Von Neumann games (cf. Section 2.3) of perfect recall can be extended to games with extended information satisfying the corresponding conditions of knowledge memory (also known as the condition of *players do not forget* in (Bonanno and Battigalli, 1997)) and action recall (also known as the condition of *players remember what choices they have made* in (Bonanno and Battigalli, 1997)) with respect to all the nodes. Let us first formally define these concepts, which we term as *memory of past knowledge* (*PKM*) (Bonanno, 2004b), and *past action recall* (*PAR*).

> (*PKM*): For all players $i \in N$ and all nodes $s, t, t' \in S$, if $s \Rightarrow^* t$ and $t \approx_i^T t'$, then there exists a node $s' \in S$ such that $s \approx_i^T s'$, and $s' \Rightarrow^* t'$.

> (*PAR*): For every player $i \in N$, for all actions $a \in \Sigma$ and for all nodes $s, t, t', x \in S$, if $s \stackrel{a}{\Rightarrow} x$, $x \Rightarrow^* t$ and $t \approx_i^T t'$, then there exist nodes $s' \in S$ and $x' \in S$ such that $s' \stackrel{a}{\Rightarrow} x'$, and $x' \Rightarrow^* t'$.

The only change in the definitions of (*PKM*) and (*PAR*) from those of (*KM*) and (*AR*), respectively, is that we have removed all mentions of S^i and replaced them by S, as the situation warrants. A significant result in this direction, proved by Bonanno (2004b), gives us that if an extensive-form game with extended information satisfies the condition (*PKM*), then it has to be a Von Neumann game. Thus, broadening our focus of perfect recall to all nodes of a game basically restricts our class of games under consideration. Nonetheless, let us now provide a sound and complete axiomatization for this restricted class of games satisfying (*PKM*) and (*PAR*). To facilitate our axiomatization, we consider an equivalent condition of (*PKM*), namely, *local memory of past knowledge* (*LPKM*), defined as follows:

> (*LPKM*): For all players $i \in N$ and all nodes $s, t, t' \in S$ if $s \Rightarrow t$ and $t \approx_i^T t'$ then there exists a node $s' \in S$ such that $s \approx_i^T s'$, and $s' \Rightarrow t'$.

Proposition 6 *In extensive-form games with extended information, PKM holds iff LPKM holds.*

To give a complete axiomatization of von Neumann games with the conditions of past knowledge memory and past action recall, we consider the following axioms.

> (A_{RootK}): **root** $\supset K_i$**root**
> (A_{LPKM}): $\langle P \rangle K_i \alpha \supset K_i \langle P \rangle \alpha$.
> (A_{PAR}): $\diamondsuit \langle \overline{a} \rangle \top \supset K_i \diamondsuit \langle \overline{a} \rangle \top$.

Theorem 7 *The complete axiom system for $\mathcal{L}$ together with (A_{RootK}), (A_{LPKM}) and (A_{PAR}) provide a complete axiom system for Von Neumann games with extended information satisfying past knowledge memory and past action recall.*

We note here that the above axiom system will not give us a sound and complete axiom system for von Neumann games satisfying perfect recall. Axioms like (A_{LPKM}) or (A_{PAR}) may not be even sound in these games. It matters how the extended information sets are defined and what properties they satisfy, as the axioms are not specific to player i nodes, in contrast to the perfect recall axioms proposed in the case of general games. We expect that once we have a complete axiomatization for Von Neumann games, adding (A_{KM}) and (A_{AR}) to the axiom system will give us a complete axiomatization for Von Neumann games satisfying perfect recall. These are open problems that we would like to tackle in the future.

5 Conclusion

As the title suggests, we have revisited the realm of dynamic games of complete information with the well-known tools from the framework of the dynamic-epistemic logics at hand. We have provided some completeness results and a decidability result as a precursor to the study of strategic reasoning from the viewpoint of players playing the games. From the players' perspectives, it is very important to be able to reason about the history of the game to come up some plan about future moves, as they do not know how the future of the game will take shape, what their opponents' moves will be, and similar other issues regarding the game. Since we are reasoning *in* games, we have not included common knowledge operator in the syntax, but we believe that by adding such an operator, the logic would still be complete and decidable in the case of general dynamic games.

With respect to the expressivity of this logic, the dynamic games discussed here can be considered as special cases of concurrent games defined in (Belardinelli, 2014; Berthon et al., 2017), but the syntax there consists of temporal and first order operators, in contrast to the dynamic operators we have here. In (Ågotnes and Alechina, 2012), players' long term powers are considered rather than individual moves. Our approach here is similar to the approach taken by Ramanujam and Simon (2008), but they consider perfect information games only.

Given these game models, a natural question to ask is that what will be a suitable notion of equivalence in these games. Is it the simple knowledge-level and action-level bisimulation for these games, or can we look into some coarser path-level equivalence? It would be interesting to have a systemic study of the fragments of the proposed language that capture the different reasonable notions of game equivalences that one can come up with. The principles of equivalence between these games as suggested by the transformations in (Thompson, 1952) and subsequent works in this direction do not take into consideration the framing effects of the games (Osborne and Rubinstein, 1994), and to the best of our knowledge, a proper notion of equivalence considering these effects is not yet available in the literature. A notion of game equivalence based on the structural details of these games will provide an answer to the problem.

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A Proof of Theorem 1

The proof of soundness is quite straightforward and hence we do not go into the details. To show completeness, we prove that every consistent formula is satisfiable. Let α_0 be a consistent formula, and let $\mathfrak{M}$ denote the set of all maximal consistent sets (MCS). We use $\mathfrak{m}, \mathfrak{m}'$ to range over MCSs. Since α_0 is consistent, there exists an MCS $\mathfrak{m}_0$ such that $\alpha_0 \in \mathfrak{m}_0$.

We now construct an extensive-form game tree model with extended information for α_0. We consider $\mathfrak{M}$ as the set of states in such a model. We first define a binary relation on MCSs as follows: $\mathfrak{m} \simeq_i \mathfrak{m}'$ iff $\{L_i\alpha | \alpha \in \mathfrak{m}'\} \subseteq \mathfrak{m}$. By axioms (A5), we have that the relation $\simeq_i$ is an equivalence relation for each $i \in N$. We also define a transition relation on MCSs as follows: $\mathfrak{m} \xrightarrow{a} \mathfrak{m}'$ iff $\{\langle a\rangle\alpha | \alpha \in \mathfrak{m}'\} \subseteq \mathfrak{m}$. Our first task will be to find a root in the model. To this end, we will now work with sets of subformulas of α_0. For a formula α, let $CL(\alpha)$ denote the closure of α under subformulas. In addition to the usual downward closure and closure under negation, we also require that **leaf**, $\diamondminus$ **root** $\in CL(\alpha)$. Let $AT(\alpha_0)$ denote the set of all maximal consistent subsets of $CL(\alpha_0)$, referred to as atoms. Because each $\mathfrak{t} \in AT(\alpha_0)$ is a finite set of formulas, we can denote the conjunction of all formulas in $\mathfrak{t}$ by $\widehat{\mathfrak{t}}$. For a nonempty subset $X \subseteq AT(\alpha_0)$, we denote by $\widetilde{X}$ the disjunction of all $\widehat{\mathfrak{t}}, \mathfrak{t} \in X$. Define a transition relation on $AT(\alpha_0)$ as follows: $\mathfrak{t} \xrightarrow{a}_{AT} \mathfrak{t}'$ iff $\widehat{\mathfrak{t}} \wedge \langle a\rangle\widehat{\mathfrak{t}'}$ is consistent. Call an atom t a *root atom* if there does not exist any atom t' such that $t' \xrightarrow{a} t$ for any a. Note that $\mathfrak{t}_0 = \mathfrak{m}_0 \cap CL(\alpha_0) \in AT(\alpha_0)$. We now prove the existence of a root atom in the atom graph followed by other useful properties of the atom graph.

Lemma 8 *There exist* $\mathfrak{t}_1, \ldots, \mathfrak{t}_k \in AT(\alpha_0)$ *and* $a_1, \ldots a_k \in \Sigma$ *(*$k \geq 0$*) such that* $\mathfrak{t}_k \xrightarrow{a_k}_{AT} \mathfrak{t}_{k-1} \ldots \xrightarrow{a_1}_{AT} \mathfrak{t}_0$*, where* $\mathfrak{t}_k$ *is a root atom.*

Proof: Consider the least set R containing $\mathfrak{t}_0$ and closed under the following condition: if $\mathfrak{t}_1 \in R$ and for some $a \in \Sigma$ there exists $\mathfrak{t}_2$ such that $\mathfrak{t}_2 \xrightarrow{a}_{AT} \mathfrak{t}_1$, then $\mathfrak{t}_2 \in R$. Now, if there exists an atom $\mathfrak{t}' \in R$ such that $\mathfrak{t}'$ is a root atom, then we are done. Suppose not, then we have $\vdash \widetilde{R} \supset \neg$**root**. But then we can show that $\vdash \widetilde{R} \supset [P]\widetilde{R}$. By rule *Past* we get $\vdash \widetilde{R} \supset \boxminus\neg$**root**. But then $\mathfrak{t}_0 \in R$ and hence $\vdash \widehat{\mathfrak{t}_0} \supset \widetilde{R}$ and therefore we get $\vdash \widehat{\mathfrak{t}_0} \supset \boxminus\neg$**root**. Since $\neg\diamondminus$**root** $\in CL(\alpha_0)$, we can deduce that $\neg\diamondminus$**root** $\in \mathfrak{t}_0$. Also, one can show that if $\vdash \widehat{\mathfrak{t}} \supset \alpha$, then $\widehat{\mathfrak{t}} \wedge \alpha$ is consistent. From axiom (A4(a)) and the fact that $\diamondminus$ **root** $\in CL(\alpha)$, we have $\diamondminus$**root** $\in \mathfrak{t}_0$, contradicting the consistency of $\mathfrak{t}_0$. □

Lemma 9 *For any two atoms* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$*, the following statements are equivalent.*

1. $\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2}$ *is consistent.*
2. $\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2}$ *is consistent.*

Proof: First, let $\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2}$ be consistent. Then we have that $[a]\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2}$ is consistent [by axiom (A3(a))], implying that $\langle a\rangle(\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2})$ is consistent, implying that $\nvdash [a]\neg(\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2})$. Therefore, $\nvdash \neg(\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2})$ [by rule (*GF*)], and hence $\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2}$ is consistent.

Conversely, let $\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge \widehat{\mathfrak{t}_2}$ be consistent. Then $\langle\overline{a}\rangle\widehat{\mathfrak{t}_1} \wedge [\overline{a}]\langle a\rangle\widehat{\mathfrak{t}_2}$ is consistent [by axiom (A3(b))], implying that $\langle\overline{a}\rangle(\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2})$ is consistent, implying that $\nvdash [\overline{a}]\neg(\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2})$, implying that $\nvdash \neg(\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2})$ [by rule (*GP*)], implying that $\widehat{\mathfrak{t}_1} \wedge \langle a\rangle\widehat{\mathfrak{t}_2}$ is consistent. □

Lemma 10 *Consider the path* $\mathfrak{t}_k \xrightarrow{a_k}_{AT} \mathfrak{t}_{k-1} \ldots \xrightarrow{a_1}_{AT} \mathfrak{t}_0$ *where* $\mathfrak{t}_k$ *is a root atom,*

1. *For all* $j \in \{0, \ldots, k-1\}$*, if* $[\overline{a}]\alpha \in \mathfrak{t}_j$ *and* $\mathfrak{t}_{j+1} \xrightarrow{a}_{AT} \mathfrak{t}_j$ *then* $\alpha \in \mathfrak{t}_{j+1}$*.*
2. *For all* $j \in \{0, \ldots, k-1\}$*, if* $\langle \overline{a} \rangle \alpha \in \mathfrak{t}_j$ *and* $\mathfrak{t}_{j+1} \xrightarrow{b}_{AT} \mathfrak{t}_j$ *then* $b = a$ *and* $\alpha \in \mathfrak{t}_{j+1}$*.*
3. *For all* $j \in \{0, \ldots, k-1\}$*, if* $\diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_j$ *then there exists* $i : j \leq i \leq k$ *such that* $\alpha \in \mathfrak{t}_i$*.*

Proof: (1) Since $\mathfrak{t}_{j+1} \xrightarrow{a}_{AT} \mathfrak{t}_j$, we have $\widehat{\mathfrak{t}_{j+1}} \wedge \langle a \rangle \widehat{\mathfrak{t}_j}$ is consistent. Since $\mathfrak{t}_{j+1}$ and $\mathfrak{t}_j$ are atoms, this is equivalent to saying that $\widehat{\mathfrak{t}_j} \wedge \langle \overline{a} \rangle \widehat{\mathfrak{t}_{j+1}}$ is consistent (by Lemma 9), which implies $[\overline{a}]\alpha \wedge \langle \overline{a} \rangle \widehat{\mathfrak{t}_{j+1}}$ is consistent (by omitting some conjuncts). Therefore $\langle \overline{a} \rangle (\alpha \wedge \widehat{\mathfrak{t}_{j+1}})$ is consistent. Using (*GP*), we get that $\alpha \wedge \widehat{\mathfrak{t}_{j+1}}$ is consistent and since $\mathfrak{t}_{j+1}$ is an atom, we have $\alpha \in \mathfrak{t}_{j+1}$.

(2) Suppose $\mathfrak{t}_{j+1} \xrightarrow{b}_{AT} \mathfrak{t}_j$, we first show that $b = a$. Suppose this is not true, since $\mathfrak{t}_{j+1} \xrightarrow{b}_{AT} \mathfrak{t}_j$, we have $\widehat{\mathfrak{t}_j} \wedge \langle \overline{b} \rangle \widehat{\mathfrak{t}_{j+1}}$ is consistent. And therefore $\widehat{\mathfrak{t}_j} \wedge \langle \overline{b} \rangle \top$ is consistent. From axiom (A2(c)), $\widehat{\mathfrak{t}_j} \wedge [\overline{a}]\bot$ is consistent. If $\langle \overline{a} \rangle \alpha \in \mathfrak{t}_j$, then we get that $\langle \overline{a} \rangle \alpha \wedge [\overline{a}]\bot$ is consistent. Therefore $\langle \overline{a} \rangle (\alpha \wedge \bot)$ is consistent. From (*GP*) we have $\alpha \wedge \bot$ is consistent, contradicting the consistency of α (ensured by the fact that $\langle a \rangle \alpha \in \mathfrak{t}_j$ is consistent).

To show that $\alpha \in \mathfrak{t}_{j+1}$ observe that $\langle \overline{a} \rangle \alpha \in \mathfrak{t}_j$ implies $[\overline{a}]\alpha \in \mathfrak{t}_j$ (by axiom (A2(b)) and closure condition). By the previous argument, we get $\alpha \in \mathfrak{t}_{j+1}$.

(3) Suppose $\diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_j$ and $\mathfrak{t}_{j+1} \xrightarrow{a}_{AT} \mathfrak{t}_j$. If $\alpha \in \mathfrak{t}_j$, then we are done. Else, by axiom (A4(b)) and the previous argument, we have $\langle \overline{a} \rangle \diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_j$. From (2), we have $\diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_{j+1}$. Continuing in this manner, we either get an i where $\alpha \in \mathfrak{t}_i$ (in which case we are done) or we get $\diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_k$. Since $\mathfrak{t}_k$ is the root atom, we have that $\widehat{\mathfrak{t}_k} \wedge \neg \langle P \rangle \top$ is consistent. Since $\diamond\!\!\!\cdot\, \alpha \in \mathfrak{t}_k$, we get that $\widehat{\mathfrak{t}_k} \wedge (\alpha \vee \langle P \rangle \alpha)$ is consistent. Thus we have that $\widehat{\mathfrak{t}_k} \wedge \alpha$ is consistent and therefore $\alpha \in \mathfrak{t}_k$. □

Canonical model construction: We are now ready to define the model M as follows. From Lemma 8 and Lemma 10 it follows that there exist MCSs $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathfrak{M}$ and $a_1, \ldots a_k \in \Sigma$ ($k \geq 0$) such that $\mathfrak{m}_k \xrightarrow{a_k}_{\mathfrak{m}} \mathfrak{m}_{k-1} \ldots \xrightarrow{a_1}_{\mathfrak{m}} \mathfrak{m}_0$, where $\mathfrak{m}_j \cap CL(\alpha_0) = \mathfrak{t}_j$. Now this path defines a (finite) tree $T_0 = (S_0, \Rightarrow_0, s_0, \widehat{\lambda})$ rooted at s_0, where $S_0 = \{s_0, s_1, \ldots, s_k\}$ and for all $j \in \{0, \cdots, k\}$, s_j is labelled by the MCS $\mathfrak{m}_{k-j}$. The relation $\Rightarrow_0$ is defined in the obvious manner. From now on, we will simply say $\alpha \in s$ where s is the tree node, to mean that $\alpha \in \mathfrak{m}$ where $\mathfrak{m}$ is the MCS associated with node s. The turn function for the non-terminal nodes (that is, for the nodes not containing the propositional atom **leaf**) is defined as expected: $\widehat{\lambda}(s_j) = i$ if $\mathbf{turn}_i \in s_j$, else $\widehat{\lambda}(s_j) = \overline{\imath}$.

As inductive hypothesis, assume that we have a tree $T_k = (S_k, \Rightarrow_k, s_0, \widehat{\lambda}_k)$. Pick a node $s \in S_k$ such that $\langle a \rangle \top \in s$ but there is no $s' \in S_k$ such that $s \overset{a}{\Rightarrow} s'$. Now, if $\mathfrak{m}$ is the MCS associated with node s, there exists an MCS $\mathfrak{m}'$ such that $\mathfrak{m} \xrightarrow{a}_{\mathfrak{m}} \mathfrak{m}'$. Pick a new node $s' \notin S_k$ and define $T_{k+1} = (S_{k+1}, \Rightarrow_{k+1}, s_0, \widehat{\lambda}_k)$ where $S_{k+1} = S_k \cup \{s'\}$ and $\Rightarrow_{k+1} = \Rightarrow_k \cup \{(s, a, s')\}$, where $\mathfrak{m}'$ is the MCS associated with s'. One can show that every node in T_{k+1} has witnesses for past formulas as well. The turn function is extended as defined earlier for the newly added nodes.

Now consider $T = (S, \Rightarrow, s_0, \widehat{\lambda})$ defined by: $S = \bigcup_{k \geq 0} S_k$ and $\Rightarrow = \bigcup_{k \geq 0} \Rightarrow_k$. Let $\Rightarrow^*$ denote the reflexive and transitive closure of the $\Rightarrow$ relation. For each $i \in N$, let us take the relation $\approx_i^T$ on S to be the restriction of $\simeq_i$ on S. Then $\approx_i^T$ forms an equivalence relation on S. We now need to ensure that the extensive-form game tree T constructed in this way has witnesses for past and future formulas as well as knowledge formulas. Define the model $M = (T, \{\sim_i^T\}_{i \in N}, \{\approx_i^T\}_{i \in N}, V)$ where for each $i \in N$, $\sim_i^T$ is the restriction of $\approx_i^T$ to S^i, the set of all player i nodes in T and $V(s) = w \cap P$, w is the MCS associated with s. Most of the conditions in the following lemma can be shown using standard modal logic and dynamic logic techniques. The only exception is the condition for $L_i\alpha$ (10), which uses the axiom (A6)(b). We give a proof below.

Lemma 11 *For any* $s \in S$*, we have the following properties.*

1. *If* $[a]\alpha \in s$ *and* $s \overset{a}{\Rightarrow} s'$*, then* $\alpha \in s'$*.*
2. *If* $\langle a \rangle \alpha \in s$*, then there exists* s' *such that* $s \overset{a}{\Rightarrow} s'$ *and* $\alpha \in s'$*.*
3. *If* $\Box\alpha \in s$ *and* $s \Rightarrow^* s'$*, then* $\alpha \in s'$*.*
4. *If* $\Diamond\alpha \in s$*, then there exists* s' *such that* $s \Rightarrow^* s'$ *and* $\alpha \in s'$*.*

5. *If $[\overline{a}]\alpha \in s$ and $s' \stackrel{a}{\Rightarrow} s$, then $\alpha \in s'$.*
6. *If $\langle\overline{a}\rangle\alpha \in s$, then there exists s' such that $s' \stackrel{a}{\Rightarrow} s$ and $\alpha \in s'$.*
7. *If $\boxminus\alpha \in s$ and $s' \Rightarrow^* s$, then $\alpha \in s'$.*
8. *If $\diamondminus\alpha \in s$, then there exists s' such that $s' \Rightarrow^* s$ and $\alpha \in s'$.*
9. *If $K_i\alpha \in s$ and $s' \approx_i^T s$, then $\alpha \in s'$.*
10. *If $L_i\alpha \in s$, then there exists s' such that $s' \approx_i^T s$ and $\alpha \in s'$.*

Proof: We now give a proof for 10, leaving the other properties to the reader. Suppose that $L_i\alpha \in s$ and $\mathfrak{m}$ be the maximal consistent set associated with s. Then by a standard modal logic argument we have that there exists $\mathfrak{m}'$ such that $\mathfrak{m}' \simeq_i \mathfrak{m}$ and $\alpha \in \mathfrak{m}'$. But this does not guarantee that $\mathfrak{m}'$ is associated with any member s' of S. We now prove that this $\mathfrak{m}'$ is indeed a member of the canonical model constructed. Then by the definition of $\approx_i^T$ we will have that $\mathfrak{m}' \approx_i^T \mathfrak{m}$, which will give us the result. In the following, we will denote this $\mathfrak{m}'$ by s'. We have $s \simeq_i s'$. To show that s' is a member of the canonical tree model we need to find an MCS t such that $t \Rightarrow^* s$ and $t \Rightarrow^* s'$. This will guarantee that s' will be a member of the tree because of the following reason: If there is some t such that $t \Rightarrow^* s$, then t will be a member of the tree containing s (by axiom (A2(b)). Now, since $t \Rightarrow^* s'$, s' will be also be a member of the same tree (by axiom (A2(a)). We now prove the existence of such a node t.

Let $w = \{\Diamond\alpha : \alpha \in s\} \cup \{\boxminus\beta : \boxminus\beta \in s'\}$. We need to show that w is consistent. Suppose not. Then there exists $w_0 = \{\Diamond\alpha_1, \ldots, \Diamond\alpha_m\} \cup \{\boxminus\beta_1, \ldots, \boxminus\beta_n\}$, where $\{\Diamond\alpha_1, \ldots, \Diamond\alpha_m\} \subseteq \{\Diamond\alpha : \alpha \in s\}$, and $\{\boxminus\beta_1, \ldots, \boxminus\beta_n\} \subseteq \{\boxminus\beta : \boxminus\beta \in s'\}$, and w_0 is inconsistent. Let $\alpha = \wedge\alpha_i$, and $\beta = \wedge\beta_j$. Since w_0 is inconsistent, we have $\vdash \Diamond\alpha \supset \neg\boxminus\beta$. Then, $\vdash \diamondminus\Diamond\alpha \supset \diamondminus\neg\boxminus\beta$. Hence, by Axiom (A6)(b), $\vdash L_i\alpha \supset \diamondminus\neg\boxminus\beta$. Now, because $\alpha \in s$, and $s \simeq_i s'$, we also have $L_i\alpha \in s'$. So, $\diamondminus\neg\boxminus\beta \in s'$, which implies $\neg\boxminus\boxminus\beta \in s'$, contradicting the consistency of s'. Thus, w is a consistent set of formulas, which can be extended to an MCS, giving our required t satisfying the conditions $t \Rightarrow^* s$ and $t \Rightarrow^* s'$, by construction. □

Lemma 12 *For all $\alpha \in \mathcal{L}$, for all $s \in S$, $\alpha \in s$ iff $M, s \models \psi$.*

Proof: This follows from Lemma 11 using an inductive argument. □

We are now ready to prove the following theorem which asserts that the axiom system is complete.

Proposition 13 *For any formula α_0, if α_0 is consistent then α_0 is satisfiable.*

Proof: Suppose α_0 is a consistent formula, then $\{\alpha_0\}$ can be extended to a maximal consistent set $\mathfrak{m}_0$. By the construction of the model $M = (T, \{\sim_i^T\}_{i\in N}, \{\approx_i^T\}_{i\in N}, V)$, there exists a node s in T such that s is labelled with $\mathfrak{m}_0$. By Lemma 12, $M, s \models \alpha_0$ and therefore α_0 is satisfiable. □

B Proof of Theorem 2

To address the satisfiability problem we define the following preliminary concepts and notations.

Definition 1 *Let $\alpha \in \mathcal{L}$ be a formula,*

1. *$ECL(\alpha) ::= CL(\alpha) \cup \mathcal{E}$ is the extended subformula closure of α where*
   $CL(\alpha)$ is the standard downward closure and negation closure of α and
   $\mathcal{E} ::= \{\mathbf{turn}_i \mid i \in N\} \cup \{\langle a\rangle\top,\ \langle\overline{a}\rangle\top \mid a \in \Sigma\} \cup \{\mathbf{root}, \mathbf{leaf}\}$.
2. *Any subset $\mathfrak{t} \subseteq ECL(\alpha)$ is said to be an atom if there is some maximal consistent set $\mathfrak{m}$ such that $\mathfrak{t} = ECL(\phi) \cap m$.*
   Let $AT(\alpha) ::= \{\mathfrak{t} \subseteq ECL(\alpha) \mid \mathfrak{t}$ is an atom $\}$.

3. $\mathsf{A} \subseteq AT(\alpha)$ *is said to be an* "i-admissible atom set of α" *if*

   - *If there is some* $\mathsf{t} \in \mathsf{A}$ *such that* $\mathbf{turn}_i \in \mathsf{t}$ *then for all* $\mathsf{t}' \in \mathsf{A}, \mathbf{turn}_i \in \mathsf{t}'$.
   - *If there is some* $\mathsf{t} \in \mathsf{A}$ *and* $K_i\psi \in \mathsf{t}$ *then for all* $\mathsf{t}' \in \mathsf{A}$ $\psi \in \mathsf{t}'$.
   - *For all* $\mathsf{t} \in \mathsf{A}$ *and for all* $L_i\psi \in \mathsf{t}$ *there is some* $\mathsf{t}' \in \mathsf{A}$ *such that* $\psi \in \mathsf{t}'$.

   Let $\mathsf{AS}^i(\alpha) = \{\mathsf{A} \subseteq AT(\alpha) \mid \mathsf{A}$ *is an i-admissible atom set of* $\alpha\}$ *and* $\mathsf{AS}(\alpha) = \bigcup_i(\mathsf{AS}^i)$.

4. *An atom graph of α is given by* $G(\alpha) = (V, E, \pi, (\mathsf{B}^i)_{i \in N})$ *where*

   - *V is a finite set of nodes.*
   - *The map* $\pi : V \to AT(\alpha)$ *associates every node of the graph to some atom such that there exists a unique "root vertex"* $r \in V$ *such that* $\mathbf{root} \in \pi(r)$ *and for all* $v \neq r$ *we have* $\mathbf{root} \notin \pi(v)$. *Also there is some* $w \in V$ *such that* $\alpha \in \pi(w)$.
   - $E \subseteq (V \times \Sigma \times V)$ *is the labelled directed edge set. For every vertex* $v \in V$ *define* $Suc(v) = \{u \mid$ *for some* $a \in \Sigma, (v, a, u) \in E\}$ *and* $Pred(v) = \{u' \mid$ *for some* $a \in \Sigma, (u', a, v) \in E\}$. *Also let* $In(v) = \{a \mid$ *for some* $u \in V, (u, a, v) \in E\}$ *and* $Out(v) = \{a \mid$ *for some* $u' \in V, (v, a, u') \in E\}$. *E satisfies the following conditions:*
     - $Pred(r) = \emptyset$ *and all* $v \in V$ *are reachable from* r.
     - *For all* $v \neq r$, $In(v)$ *is singleton.*
     - *If* $(u, a, v) \in E$ *then* $\{\psi \mid [a]\psi \in \pi(u)\} \cup \{\Box\psi, \psi \mid \Box\psi \in \pi(u)\} \subseteq \pi(v)$ *and* $\{\psi \mid [\overline{a}]\psi \in \pi(v)\} \cup \{\boxminus\psi, \psi \mid \boxminus\psi \in \pi(u)\} \subseteq \pi(v)$.
   - *For all* $i \in N$, $\mathsf{B}^i = \{\mathsf{B}^i_0, \mathsf{B}^i_1, \cdots, \mathsf{B}^i_{i_n}\}$ *is a partition over V such that for every* $j \leq i_n$ *the set* $\pi(\mathsf{B}^i_j) = \{\pi(v) \mid v \in \mathsf{B}^i_j\}$ *is an i-admissible atom set.*
   - *For all* $v \in V$ *and for all* $i \in N$ *let* $V^i = \{v \mid \mathbf{turn}_i \in \pi(v)\}$. *Now define* $B^i(v) = B^i_j$ *such that* $v \in B^i_j$. *We have the condition that for all* $i \in N$ *the set* $\{B^i(v) \mid v \in V^i\}$ *forms a partition over* V^i.

5. *Given an atom graph,* $G(\alpha) = (V, E, \pi, (\mathsf{B}^i)_{i \in N})$, *any* $v \in V$ *is said to be saturated if the following holds:*

   - *If* $\langle\overline{a}\rangle\psi \in \pi(v)$ *then there is some u such that* $u \xrightarrow{a} v$ *and* $\psi \in \pi(u)$.
   - *If* $\langle a\rangle\psi \in \pi(v)$ *then there is some* $u \in V$ *such that* $v \xrightarrow{a} u$ *and* $\psi \in \pi(u)$.
   - *If* $\diamondminus\psi \in \pi(v)$ *then for all paths* $r \xrightarrow{a_0} v_1 \xrightarrow{a_1} \cdots v_l = v \in P_v$ *there is some* $j \leq l$ *such that* $\psi \in \pi(v_j)$.
   - *If* $\Diamond\psi \in \pi(v)$ *then there is some path* $v \xrightarrow{b_0} v_1 \xrightarrow{b_1} \cdots \xrightarrow{b_k} v_k$ *such that* $\psi \in \pi(v_k)$.

   $G(\alpha)$ is saturated if all $v \in V$ are saturated.

Definition 2 *Let* $\hat{T_\mathcal{I}} = (T_\mathcal{I}, \{\approx^T_i\}_{i \in N})$ *be an extensive-form game with extended information, where* $T_\mathcal{I} = (T, \{\sim^T_i\}_{i \in N})$ *is an extensive-form game with imperfect information with* $T = (S, \Rightarrow, s_0, \widehat{\lambda})$. *For every* $i \in N$, *let* $Part_i ::= \{S^i_1, S^i_2, \cdots\}$ *be the partition induced by* $\sim^T_i$ *over* S^i. *and we have an induced partition* $\{S^i_j \mid i \in N$ *and* $S^i_j \in Part_i\}$ *over S. Consider a model* $M = (T, \{\sim^T_i\}_{i \in N}, \{\approx^T_i\}_{i \in N}, V)$. *Let α be a formula such that for some* $w \in S$, *we have* $M, w \models \alpha$. *Define the following:*

1. *For every* $s \in S$, *the set* $atom(s) = ECL(\alpha) \cap \{\psi \mid M, s \models \psi\}$ *denotes the set of formulas that are true at s restricted to* $ECL(\alpha)$.

2. *For every* $i \in N$ *and for every* $S^i_j \in Part_i$, *the set* $bag(S^i_j) = \{atom(s) \mid s \in S^i_j\}$ *denotes the set of all atoms corresponding to the information set* S^i_j.

3. *For all* $i \in N$, *and* $S^i_j, S^i_k \in Part_i$, *define* $S^i_j \eqsim_i S^i_k$ *if* $bag(S^i_j) = bag(S^i_k)$.

4. *For all $i \in N$ and $s \in S$ define $bag^i(s) = S^i_j$ such that $s \in S^i_j$.*

5. *For all $s, t \in S$ define $s \eqsim t$ if $atom(s) = atom(t)$ and for all $i \in N$, $bag^i(s) = bag^i(t)$.*

Note that for all $i \in N$, $\eqsim_i$ is an equivalence relation over $Part_i$. Also, since $bag(S^i_j) \subseteq ECL(\phi)$, the equivalence relation $\eqsim_i$ partitions $Part_i$ into at most $2^{(O(|\alpha|)}$ sets. Similarly, $\eqsim$ is an equivalence relation over S and the size of the partition is bounded by $2^{(O|\alpha|)}$.

Proposition 14 *Any formula $\alpha \in \mathcal{L}$, α is satisfiable iff there is a saturated atom graph $G(\alpha)$.*

Proof: ($\Leftarrow$). Let $G(\alpha) = (V, E, \pi, (B^i)_{i \in N})$ be a saturated atom graph of α with root $r \in V$. Define the unravelling of $G(\alpha)$ starting from r given by $\overline{G}(\alpha) = (W, R)$ where W is the set of all finite paths in G starting from r and $\bar{v}u \xrightarrow{a} \bar{v}uu' \in R$ if $u \xrightarrow{a} u'$.

Define the tree model $\hat{T_{\mathcal{I}}} = (T_{\mathcal{I}}, \{\approx^T_i\}_{i \in N})$ where $T_{\mathcal{I}} = (T, \{\sim^T_i\}_{i \in N})$ with $T = (W, R, r, \widehat{\lambda})$ as follows:

- For all $\bar{v}u \in W$, $\widehat{\lambda}(\bar{v}u) = i$ iff $\mathbf{turn}_i \in \pi(u)$.
- For all $\bar{u}u', \bar{v}v' \in W$, $\bar{u}u' \approx_i \bar{v}v'$ if there is some $j \leq i_n$ such that $\{u', v'\} \subseteq B^i_j$. Two tree nodes are indistinguishable for player i if the last nodes of the corresponding paths belong to the same partition with respect to B^i (cf. Definition 1.4).
- For all $\bar{u}u', \bar{v}v' \in W^i$, the set of all paths ending with player i nodes, as given by $\widehat{\lambda}$ defined above, $\bar{u}u' \sim^T_i \bar{v}v'$ if $B^i(u') = B^i(v')$.

Note that the last condition in Definition 1.4 ensures that $\sim^T_i$ is a restriction of $\approx_i$ over W^i. Hence $T_{\mathcal{I}}$ is well defined. Now define the model corresponding to $\hat{T_{\mathcal{I}}}$ given by $M = (\hat{T_{\mathcal{I}}}, V)$ where for all $\bar{v}u \in W, p \in V(\bar{v}u)$ iff $p \in \pi(u)$. We have the truth lemma.

Claim. For all $\bar{v}u \in W$ and for all $\psi \in ECL(\phi)$, $\psi \in \pi(u)$ iff $M, \bar{v}u \models \psi$.

The proof is by induction on the structure of ψ. The base case of propositions follows from the definition of V. The $\neg\psi$ and $\psi_1 \wedge \psi_2$ cases are routine.

The case of $\langle a \rangle\psi$, $\langle \overline{a} \rangle\psi$, $\Diamond\psi$ and $\diamondminus\psi$ follows from the fact that $G(\alpha)$ is saturated. We verify it for $\diamondminus\psi$ case. Suppose $\diamondminus\psi \in \pi(u)$. Let $\bar{v}u = v_1 v_2 \cdots v_n$. Now this is a path from $r (= v_1)$ to $u (= v_n)$ and since $G(\alpha)$ is saturated, there is some v_i such that $\psi \in \pi(v_i)$ and by induction hypothesis $M, (v_1 \cdots v_i) \models \psi$.

If $M, \bar{v}u \models \diamondminus\psi$, let $M, (v_1 \cdots v_i) \models \psi$. Then by induction hypothesis, $\psi \in \pi(v_i)$. This means there is at least one path from r to u such that there is some v_i on that path where $\psi \in v_i$. Now suppose $\diamondminus\psi \notin \pi(u)$ then since $\pi(u)$ is an atom we have $\boxminus\neg\psi \in \pi(u)$ and since $G(\alpha)$ is an atom graph, for all ancestors v of u we have $\neg\psi \in \pi(v)$ which contradicts to the observation.

For $L_k\psi$ case, suppose $L_k\psi \in \pi(u)$. Let $u \in B^k_j$. Then, since B^k_j is a k-admissible atom, there is some $w \in B^k_j$ such that $\psi \in \pi(w)$. Since all nodes are reachable from r, let $rw_1 \cdots w_n w$ be some path from r to w. Thus, by definition of $\approx_k$ we have $(rw_1 \cdots w_n w) \approx_k \bar{v}u$ and by induction hypothesis $M, (rw_1 \cdots w_n w) \models \psi$. Hence $M, \bar{v}u \models L_k\psi$.

Suppose $M, \bar{v}u \models L_k\psi$. Then there is some $\bar{v'}u' \approx_k \bar{v}u$ such that $M, \bar{v'}u' \models \psi$ and by induction hypothesis $\psi \in \pi(u')$. By construction, it has to be the case that $B^k(u) = B^k(u')$. Now suppose $L_k\psi \notin \pi(u)$ then $K_k\neg\psi \in \pi(u)$ and since $B^k(u)$ is k-admissible atom we have $\neg\psi \in \pi(u')$ which is a contradiction.

($\Rightarrow$). Let $M = (T, \{\sim^T_i\}_{i \in N}, \{\approx^T_i\}_{i \in N}, V)$ be a model with $T = (S, \Rightarrow, s_0, \widehat{\lambda})$ such that for some $w \in S$ we have $M, w \models \alpha$.

We have defined $s \eqsim t$ in Definition 2.4 which has a bounded index. Let $[s] = \{t \mid s \eqsim t\}$. Define the atom graph $G(\alpha) = (W, E, \pi, (B^i)_{i \in N})$ where

- $W = \{[s] \mid s \in S\}$.
- $[s] \xrightarrow{a} [t] \in E$ if there is some $s_1 \in [s]$ and $t_1 \in [t]$ such that $s_1 \xrightarrow{a} t_1 \in \Rightarrow$.
- For all $[s] \in W$, define $\pi([s]) = atom(s)$.

- Define B^i to be the partition induced by the relation $\triangleq_i$ over W where,
  $[s] \triangleq_i [t]$ if $bag^i(s) = bag^i(t)$.

Note that π is well defined, since for all $s_1, s_2 \in [s]$ we have $atom(s_1) = atom(s_2)$. Similarly $\triangleq_i$ is well defined, since for all $s_1, s_2 \in [s]$ we have $bag^i(s_1) = bag^i(s_2)$.

First we verify that $G(\alpha)$ is an atom graph. Clearly W is finite and $\pi : W \to AT(\alpha)$ is well defined. Since there is a unique $r(= s_0) \in S$ such that **root** $\in atom(r)$, we have $[r] \in W$. Further for all $t \neq r$ we have $M, t \not\models$ **root** and hence **root** $\notin \pi([t])$. Also since $M, w \models \alpha$ we have a $[w] \in W$ such that $\alpha \in \pi([w])$.

Now we verify all the conditions of E. Since $[r] = \{r\}$ we have $Pred([r]) = \emptyset$. Also, for any $[s] \in W$, let $r \xrightarrow{a_0} t_1 \cdots t_n \xrightarrow{a_n} s$ be the path in the tree from the root to s. Then, by construction, $[r] \xrightarrow{a_0} [t_1] \cdots [t_n] \xrightarrow{a_n} [s]$ is a path in $G(\alpha)$ and hence all nodes are reachable from $[r]$.

Suppose $In([s]) \geq 2$. Then there exist $t_1, t_2 \in S$ and $s_1, s_2 \in [s]$ such that $t_1 \xrightarrow{a} s_1 \in \Rightarrow$ and $t_2 \xrightarrow{b} s_2 \in \Rightarrow$ for some $a \neq b$. This means $\langle\overline{a}\rangle\top \in atom(s_1)$ and $\langle\overline{b}\rangle\top \in atom(s_2)$. But, $atom(s_1) = atom(s_2)$, which implies $\{\langle\overline{a}\rangle\top, \langle\overline{b}\rangle\top\} \subset atom(s_1)$, a contradiction to the consistency of $atom(s_1)$.

Suppose $[s] \xrightarrow{a} [t]$ then there is some $s_1 \in [s]$ and $t_1 \in [t]$ such that $s_1 \xrightarrow{a} t_1$. Now it is easy to see that $\{\psi \mid [a]\psi \in \pi([s_1])\} \cup \{\Box\psi, \psi \mid \Box\psi \in \pi([s_1])\} \subseteq \pi([t_1])$ and $\{\psi \mid [\overline{a}]\psi \in \pi([t_1])\} \cup \{\boxminus\psi, \psi \mid \boxminus\psi \in \pi[t_1]\} \subseteq \pi([s_1])$.

Finally we need to check the conditions for B^i. Let $B^i_k \in B^i$. We need to prove that $\{\pi([s]) \mid [s] \in B^i_k\}$ is an i-admissible atom set. Suppose for some $[s] \in B^i_k$, we have $\mathbf{turn}_i \in \pi([s]) = atom(s)$. Hence $\widehat{\lambda}(s) = i$ which implies that $M, s \models \mathbf{turn}_i$ which means $s \in S^i$. Now for all $[t] \in W$, if $[s] \triangleq_i [t]$ then $bag^i(s) = bag^i(t)$. Thus there is some $s' \in bag^i(t)$ such that $atom(s') = atom(s)$ which implies $M, s' \models \mathbf{turn}_i$. Now since $\approx_i$ extends $\sim_i^T$ and $s' \sim_i^T t$, we have $\mathbf{turn}_i \in atom(s')$ iff $M, t \models \mathbf{turn}_i$ iff $\mathbf{turn}_i \in atom([t])$. Thus, if $\mathbf{turn}_i \in \pi([s])$ and $[s] \triangleq_i [t]$, then $\mathbf{turn}_i \in \pi([t])$.

To show that the set $\{B^i([s]) \mid [s] \in W^i\}$ is a partition over W^i it is enough observe that $\triangleq_k$ is an equivalence relation over W^i.

Finally to see that the graph is saturated we verify the condition for $\langle\overline{a}\rangle\psi$ and the rest are similar. Suppose $\langle\overline{a}\rangle\psi \in \pi([s])$. Then, $M, s \models \langle\overline{a}\rangle\psi$. Then there exists some node t such that $t \xrightarrow{a} s \in \Rightarrow$ and $M, t \models \psi$. So, $\psi \in \pi([t])$ and $[t] \xrightarrow{a} [s]$. This completes the proof.

$\square$

Corollary 15 *Satisfiability problem for $\mathcal{L}$ is in non-deterministic double exponential time.*

Proof: Suppose α is satisfiable. Then, by Proposition 14, there is a saturated atom graph whose size is at most double exponential in the length of α. Thus a non-deterministic algorithm can guess the graph and check that it is indeed saturated. $\square$

C Proof of Proposition 3

The soundness argument is straightfrward. The completeness proof follows from Theorem 1 and the following. We show that our canonical model satisfies the (*KM*) condition. Let $i \in N$, and nodes $s, t, t' \in S_i$ be such that $s \Rightarrow^* t$ and $t \sim_i^T t'$. Let $w = \{\Diamond\alpha : \alpha \in t'\} \cup \{\beta : K_i\beta \in s\}$. We want to prove that w is consistent. Suppose not. Then there exists $w_0 = \{\Diamond\alpha_1, \ldots, \Diamond\alpha_m\} \cup \{\beta_1, \ldots, \beta_n\}$, where $\{\Diamond\alpha_1, \ldots, \Diamond\alpha_m\} \subseteq \{\Diamond\alpha : \alpha \in t'\}$, and $\{\beta_1, \ldots, \beta_n\} \subseteq \{\beta : K_i\beta \in s\}$, and w_0 is inconsistent. Let $\alpha = \wedge\alpha_i$, and $\beta = \wedge\beta_j$. Since w_0 is inconsistent, we have $\vdash \Diamond\alpha \supset \neg\beta$. Then $\vdash L_i\Diamond\alpha \supset L_i\neg\beta$. Then by axiom (A_{KM}), $\vdash \Diamond(\mathbf{turn}_i \wedge L_i\alpha) \supset L_i\neg\beta$. Now, $\alpha \in t'$, implying that $L_i\alpha \in t'$, implying that $L_i\alpha \in t$, because $t \sim_i^T t'$, implying that $\mathbf{turn}_i \wedge L_i\alpha \in t$. So, $\Diamond(\mathbf{turn}_i \wedge L_i\alpha) \in s$ as $s \Rightarrow^* t$.Then $L_i\neg\beta \in s$, implying $\neg K_i\beta \in s$, contradicting the consistency of s. Thus, w is a consistent set of formulas which can be extended to an mcs giving our required s' satisfying the conditions $s \sim_i^T s'$, and $s' \Rightarrow^* t'$, by construction.

D Proof of Proposition 4

The proof follows from Theorem 1 and the corresponding characterization result given in Proposition 8 of Bonanno (2004a).

E Proof of Corollary 5

The proof follows from Propositions 3, 4 and Proposition 9 of Bonanno (2004a).

F Proof of Proposition 6

Let us first assume that *PKM* holds. Let $i \in N$ and $s, t, t' \in S$ such that $s \Rightarrow t$ and $t \approx_i^T t'$. Then, by *PKM*, we have that there exists a node $s' \in S$ such that $s \approx_i^T s'$, and $s' \Rightarrow^* t'$. Now, due to a result from (Bonanno, 2004b) we can assume that the game is von Neumann. Hence, $l(s') = l(s) = l(t) - 1 = l(t') - 1$. Thus $s' \Rightarrow t'$.

Conversely, let us assume that *LPKM* holds. Let $i \in N$ and $s, t, t' \in S$ such that $s \Rightarrow^* t$ and $t \approx_i^T t'$. We need to find $s' \in S$ such that $s \approx_i^T s'$, and $s' \Rightarrow^* t'$. If $s = t$, then $s' = t'$. If not, there exists $a_1, \ldots, a_m$, such that $s \overset{a_0}{\Rightarrow} s_1 \cdots \overset{a_{m-1}}{\Rightarrow} s_m = t$. Since $t \approx_i^T t'$ and $s_{m-1} \Rightarrow t$, there exists s'_{m-1} such that $s_{m-1} \approx_i^T s'_{m-1}$, and $s'_{m-1} \Rightarrow t'$. Proceeding in a similar way, we finally get an s' such that $s \approx_i^T s'$, and $s' \Rightarrow^* t'$.

G Proof of Theorem 7

The soundness proofs are straightforward. The completeness proof follows from Theorem 1 and the following. We first show that our canonical model satisfies the (*LPKM*) condition. Let $s, t, t' \in S$ be such that $s \Rightarrow t$ and $t \approx_i^T t'$. Let $w = \{\alpha : [P]\alpha \in t'\} \cup \{\beta : K_i\beta \in s\}$. We want to prove that w is consistent. Suppose not. Then there exists $w_0 = \{\alpha_1, \ldots, \alpha_m\} \cup \{\beta_1, \ldots, \beta_n\}$, where $\{\alpha_1, \ldots, \alpha_m\} \subseteq \{\alpha : [P]\alpha \in t'\}$, and $\{\beta_1, \ldots, \beta_n\} \subseteq \{\beta : K_i\beta \in s\}$, where w_0 is inconsistent. Let $\alpha = \wedge\alpha_i$, and $\beta = \wedge\beta_j$. Since w_0 is inconsistent, we have $\vdash \beta \supset \neg\alpha$. Then $\vdash K_i\langle P\rangle\beta \supset K_i\langle P\rangle\neg\alpha$. Then, by axiom ($A_{LPKM}$) $\vdash \langle P\rangle K_i\beta \supset K_i\langle P\rangle\neg\alpha$. Now, $K_i\beta \in s$, implying that $\langle P\rangle K_i\beta \in t$, because $s \Rightarrow t$. So, $K_i\langle P\rangle\neg\alpha \in t$, implying that $\langle P\rangle\neg\alpha \in t'$, a contradiction to the fact that $[P]\alpha \in t'$, t' being consistent. Thus, w is a consistent set of formulas which can be extended to an mcs giving our required s' satisfying the conditions $s \approx_i^T s'$ (by construction), and $s' \Rightarrow t'$, which follows from (A_{RootK}) and the derived theorem $(\langle P\rangle\alpha \wedge \langle\overline{a}\rangle\top) \supset \langle\overline{a}\rangle\alpha$.

We now show that the canonical model satisfies (A_{PAR}). Let $i \in N$, $a \in \Sigma$ and s $s, t, t', x \in S$, such that $s \overset{a}{\Rightarrow} x$, $x \Rightarrow^* t$ and $t \approx_i^T t'$. Suppose for all $s' \in S$ and $x' \in S$ if $s' \overset{a}{\Rightarrow} x'$, then $x' \not\Rightarrow^* t'$. Now, $\langle\overline{a}\rangle\top$ holds at x, and thus $\diamondminus\langle\overline{a}\rangle\top$ holds at t. By (A_{PAR}), $K_i\diamondminus\langle\overline{a}\rangle\top$ holds at t. Then $\diamondminus\langle\overline{a}\rangle\top$ holds at t'. Then there exists $x' \Rightarrow^* t'$ such that $\langle\overline{a}\rangle\top$ holds at x', contradicting our assumption. This completes the proof.