On the subtle nature of a simple logic of the hide and seek game

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Abstract. We discuss a simple logic to describe one of our favourite games from childhood, hide and seek, and show how a simple addition of an equality constant to describe the winning condition of the seeker makes our logic undecidable. There are certain decidable fragments of first-order logic which behave in a similar fashion and we add a new modal variant to that class of logics. We also discuss the relative expressive power of the proposed logic in comparison to the standard modal counterparts.

1 From games to logic

Everyone remembers the pleasure of playing hide and seek in her or his childhood. After calling out "I am ready, you can come to find me", the fun part is to stay at your secret spot, not making any noise, and to expect that the other player would not discover you. Once you are found, the other wins. Let us consider a two-player setting, use E to denote the hider, and A the seeker. Following the research program of [7], the game of hide and seek is naturally seen as a graph game, where A and E are located at two different nodes, and are allowed to move around. The goal of A is to meet E, while the goal of E is to avoid A. For the game that many of us played in childhood, the player E (one who hides) basically stays at one place, whereas player A (one who seeks) moves from one node to another. We can describe such graph games using the basic modal logic. However, if we consider a simple modification by allowing moves for both the players (akin to the game of cops and robber [23]), the setting becomes quite diverse. On one hand, these graph games are natural candidates for modelling computational search problems, on the other hand, the nuanced interaction between the players playing hide and seek is a showcase of interactive players having their goals entangled, which is a popular phenomenon in social networks. In other words, the graph game of hide and seek provides us with an ideal arena where we can study reasoning about social interaction and challenges therein arising from such intertwined objectives of players. In the following we will make these games more precise and provide a language to express strategic reasoning and winning conditions of players.

However, before going into the logic details, let us first get a feel about the hide and seek game regarding the information available to the players. That will also lead us to understand the kind of reasoning that we plan to explore for such games. Essentially, it is an imperfect information game where the seeker is not aware of the position of hider, whereas the hider may or may not know the exact position of the seeker. Both the players know the game graph where they move about and are aware of their own positions and moves. Now, the modification that we talk about makes the setting even more interesting information-wise, as then we can consider different levels of information available to both the players. However, to keep things simple we start off from a high-level modeller's perspective, that is, we reason *about* such games. Thus, we reason about players' observations and moves with the assumption that the whole graph and the players' positions at each stage of the game are available to us. We leave the players' perspectives for future work.

Coming back to the game proper, we have the two players located at two different nodes. To model their moves we consider a pair of states as an evaluation point rather than a single state in a Kripke model (a pointed model), and consider distinct modalities to express the moves of the players. The evaluation of these two different modalities, one for each player, can then be assessed coordinate-wise with respect to the pair of states. In addition, a winning condition for the hide and seek game corresponding to the seeker finding the hider can be modelled by considering a pair of states whose first and second elements are the same. This basically gives us the identity relation which can be expressed by introducing a special identity proposition. We first note that using standard modal logic arguments, one can show the decidability of the satisfaction problem of the two-dimensional modal logic mentioned above, without the special proposition. Interestingly enough, such a simple addition, viz. incorporating the identity proposition, transforms a decidable modal logic into an undecidable one. In fact, there are various elegant examples of logics that suggest that taking this identity relation into account may change previously decidable logics (without equality) into undecidable ones, e.g., the Gödel class of first-order formulas with identity (cf. [16]). A more recent example is the logic of functional dependence with function symbols (see [4] and [24]). We add one more logic to this class, and that constitutes the main technical result of this paper. This result also refutes a claim mentioned in [7] which stated that the extended logic with the identity proposition will remain decidable. The related notion of expressive power of the proposed logic is also discussed here.

We finally note that this modified version of hide and seek game played on graphs is a special case of cops and robber game [23], a classic pursuit-evasion game played on graphs, where several cops attempt to catch a robber. The hide and seek game corresponds to the game having a single cop chasing a robber. Thus, this study opens up the possibility of a logical analysis of these cops and robber games with all their generality (cf. [23]) which have been well-studied from algorithmic and combinatorial perspectives. We are currently working on this idea and exploring an extension of the logic proposed here with modal substitution operators [26].

In section 2 we introduce a logic (LHS) to reason about plays and winning conditions in the hide and seek game. Section 3 deals with the relative expressive power of the language and relevant notions of bisimulation are introduced to facilitate the discussion. Section 4 gives the main result of this work, viz. the satisfaction problem of LHS is undecidable. Section 5 provides a discussion on related work, and section 6 gives pointers to further research.

2 Logic of hide and seek (LHS)

Let us first introduce a logic to describe the game of hide and seek, LHS, followed by an informal discussion about the expressivity of the proposed logic.

Definition 1 (Language). *Let* $\mathsf{P_E}$ *denote a countable set of propositional variables for player* E*, and* $\mathsf{P_A}$ *for player* A*. The two dimensional modal language is given as follows:*

$$\varphi ::= p_A \mid p_E \mid I \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle \mathsf{left} \rangle\varphi \mid \langle \mathsf{right} \rangle\varphi$$

where $p_E \in \mathsf{P_E}$*,* $p_A \in \mathsf{P_A}$*, and* I *is a propositional constant. Other Boolean connectives are defined in the usual way, and so are the corresponding box modalities* $[\mathsf{left}]$ *and* $[\mathsf{right}]$*.*

Without loss of generality, the modal operator representing player E's moves is given by $\langle \mathsf{left} \rangle$ and that representing A's moves is given by $\langle \mathsf{right} \rangle$. Formulas are evaluated in *standard relational models* $\mathbf{M} = (W, R, \mathsf{V})$, where W is a non-empty set of vertices, $R \subseteq W \times W$ is a set of edges, and $\mathsf{V} : \mathsf{P_E} \cup \mathsf{P_A} \to 2^W$ is a valuation function. Moreover, for any $s, t \in W$, we call $(\mathbf{M}, s, t)$ a *pointed graph model for two players* (for simplicity, *graph model)*: intuitively, s and t represent respectively the positions of players E and A. To simplify notations, we also employ $\mathbf{M}, s, t$ for $(\mathbf{M}, s, t)$. Semantics for LHS is given by the following:

Definition 2 (Semantics). *Let* $\mathbf{M} = (W, R, \mathsf{V})$ *be a model and* $s, t \in W$*. Truth of formulas* φ *at the graph model* $(\mathbf{M}, s, t)$*, written as* $\mathbf{M}, s, t \vDash \varphi$*, is defined recursively as follows:*

$$\begin{array}{rcl}
\mathbf{M}, s, t \vDash p_E & \Leftrightarrow & s \in \mathsf{V}(p_E) \\
\mathbf{M}, s, t \vDash p_A & \Leftrightarrow & t \in \mathsf{V}(p_A) \\
\mathbf{M}, s, t \vDash I & \Leftrightarrow & s = t \\
\mathbf{M}, s, t \vDash \neg\varphi & \Leftrightarrow & \mathbf{M}, s, t \nvDash \varphi \\
\mathbf{M}, s, t \vDash \varphi \wedge \psi & \Leftrightarrow & \mathbf{M}, s, t \vDash \varphi \textit{ and } \mathbf{M}, s, t \vDash \psi \\
\mathbf{M}, s, t \vDash \langle \mathsf{left} \rangle\varphi & \Leftrightarrow & \exists s' \in W \textit{ s.t. } Rss' \textit{ and } \mathbf{M}, s', t \vDash \varphi \\
\mathbf{M}, s, t \vDash \langle \mathsf{right} \rangle\varphi & \Leftrightarrow & \exists t' \in W \textit{ s.t. } Rtt' \textit{ and } \mathbf{M}, s, t' \vDash \varphi
\end{array}$$

As mentioned earlier, the above language has two modalities, one for each player, viz. $\langle \mathsf{left} \rangle$ for player E and $\langle \mathsf{right} \rangle$ for player A. Accordingly, all the formulas are evaluated in a graph model. The constant I denotes the identity relation in a game graph to describe the meeting of two players, signifying the fact that the seeker has found the hider. Let us denote LHS_{-I} to be the logic LHS without the constant I.

Here are some useful notions. Given a model $\mathbf{M}$ and a set $U \subseteq W$ of states, define $R(U) := \{t \in W \mid \textit{there is } s \in U \textit{ with } Rst\}$, denoting the set of successors of the points in U. For simplicity, we usually write $R(s)$ for $R(\{s\})$ when U is a singleton $\{s\}$. We can introduce the logical notions such as *satisfiability* and *modal equivalence* in the usual way, and we will omit the details here.

Going back to the hide and seek game itself, one can consider different variants played on the game graph model, e.g., the players can move simultaneously or sequentially. In a sequential play, one can also consider different orders of play. In this paper, we assume that the players move sequentially, and that the hider E starts the game. Local one-step winning positions (pairs of states describing the current positions of the players) for each player can be expressed in our language as follows:

- E : $\langle \mathsf{left} \rangle [\mathsf{right}] \neg I$
- A : $[\mathsf{left}] \langle \mathsf{right} \rangle I$

More generally, winning positions for E and A can be described as:

- E: $\forall n (\langle \mathsf{left} \rangle [\mathsf{right}])^n \neg I$
- A: $\exists n ([\mathsf{left}] \langle \mathsf{right} \rangle)^n I$

Note that the above conditions involve countable conjunction/disjunction of finite iterations of interactions between two players. The interactions $\langle \mathsf{left} \rangle [\mathsf{right}]$/ $[\mathsf{left}] \langle \mathsf{right} \rangle$ are expressed with two separate modalities, but they are considered as a single unit. These are not expressible in our language. As mentioned in the introduction, we are currently exploring an extension of this language with modal substitution operators which would also provide a finitary way to express such countable boolean operations.

Remark 1. There are other ways to give suitable logics capturing the hide and seek game. For instance, one can replace identity constant I with C, denoting '*catching*': $\mathbf{M}, s, t \models C$ iff $R(s) \subseteq R(t)$. From the perspective of the game, constant C describes that all states accessible to the hider are accessible to the seeker as well. In contrast to I which states that the seeker has already won, C indicates that she can win in the next round. They amount to the same condition for games of perfect information: if the seeker has the ability to meet the hider she will actually do that, if she is rational. However, from a logical perspective, their interpretations are entirely different, leading to distinct expressive features. For an illustration, let us note that C can be defined as $[\mathsf{left}] \langle \mathsf{right} \rangle I$ in LHS, but I is not definable in the logic extending LHS_{-I} with C. The constant proposition C with the given interpretation is also useful in describing cop-win graphs in the cops and robber game involving a single cop [23], see more details in [26].

In the next two sections we will explore some logical properties of LHS regarding its expressiveness on one hand and satisfiability on the other hand.

3 Bisimulation and expressive power

The notion of bisimulation is an important tool for studying the expressive power of modal logics. We are now going to explore a suitable notion tailored to our logic. We usually need to be careful when introducing the conditions: on one hand, the definition should ensure that the logic cannot distinguish bisimilar models (i.e., the desired notion is strong enough), but on the other hand, it should also hold between two models whenever they cannot be distinguished by the logical language (thus, it is weak enough). In what follows, we take the standard bisimulation [9] as the benchmark and investigate the relations between expressiveness of basic modal logic M, LHS_{-I} and LHS. Let us start by comparing that for LHS_{-I} and M.

The standard bisimulation, denoted by $\underline{\leftrightarrow}^s$, provides us a semantic characterization of the expressiveness of the basic modal language. And at a first glance, the semantic design of logic LHS_{-I} is similar to that of the basic modal logic, except that we now need to consider two states simultaneously when evaluating formulas. So, is logic LHS_{-I} invariant under the standard notion? First, we provide a positive answer in the following sense:

Proposition 1. *If* $(\mathbf{M}, w) \underline{\leftrightarrow}^s (\mathbf{M}', w')$ *and* $(\mathbf{M}, v) \underline{\leftrightarrow}^s (\mathbf{M}', v')$*, then* $(\mathbf{M}, w, v)$ *and* $(\mathbf{M}', w', v')$ *satisfy the same formulas of* LHS_{-I}*.*

Proof. The proof is straightforward by applying induction on formulas of LHS_{-I}. We leave the details to the reader. □

Therefore, the standard bisimulation is strong enough to measure the expressive power of LHS_{-I}. But meanwhile, to behave properly, is it also weak enough? Unfortunately, we have the following negative result:

Proposition 2. *There are* $(\mathbf{M}, w, v)$ *and* $(\mathbf{M}', w', v')$ *s.t. they satisfy the same* LHS_{-I}*-formulas but at least one of* $(\mathbf{M}, w) \underline{\leftrightarrow}^s (\mathbf{M}', w')$*,* $(\mathbf{M}, v) \underline{\leftrightarrow}^s (\mathbf{M}', v')$ *may not hold.*[5]

Proof. It suffices to give a counterexample. Consider the models $\mathbf{M}$ and $\mathbf{M}'$ depicted in Figure 1. It holds that $(\mathbf{M}, w_1, w_2)$ and $(\mathbf{M}', v_1, v_2)$ satisfy the same LHS-formulas, but we do not have $(\mathbf{M}, w_1) \underline{\leftrightarrow}^s (\mathbf{M}', v_1)$. □

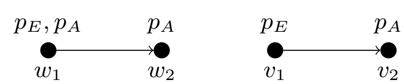

Fig. 1. Two graph models $(\mathbf{M}, w_1, w_2)$ and $(\mathbf{M}', v_1, v_2)$ satisfying same LHS-formulas.

Intuitively, the failure originates from the 'evaluation-gap' between the two worlds in our graph models $(\mathbf{M}, s, t)$: when considering atomic properties of s, both LHS_{-I} and LHS can only describe those in $\mathsf{P_E}$, but not the ones in $\mathsf{P_A}$.[6]

Now, it is time to introduce the notion of bisimulation for LHS, from which we can easily obtain that for LHS_{-I}. Here is the definition:

Definition 3 (Bisimulation for LHS models). *Let* $\mathbf{M} = (W, R, \mathsf{V})$, $\mathbf{M}' = (W', R', \mathsf{V}')$ *be two models and let* $s, t \in W$ *and* $s', t' \in W'$*. We say,* $(\mathbf{M}, s, t)$ *is bisimilar to* $(\mathbf{M}', s', t')$ *(denoted by* $(\mathbf{M}, s, t) \underline{\leftrightarrow} (\mathbf{M}', s', t')$*) if*

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[5] Strictly speaking, a negative result holds even for the basic modal logic (see [9]). However, it is still ideal if the notion of bisimulation can behave well in a large class of models (e.g., *image-finite models*). This is also one of our guiding spirits. But, as illustrated by the counterexample used to show the result, the standard notion even excludes situations that are very simple but cannot be distinguished by LHS_{-I}.

[6] From the perspective of games, the evaluation-gap suggests a way to handle situations where the two players have different observations even when they are at the same position. For example, the gap might allow us to consider further enrichments so that the states in the playing arena can encode different properties for the players: a crowed street reducing the possible moves of the escaping robber is helpful for a chasing cop, meanwhile, it is definitely a disaster to the robber.

Atom: $(\mathbf{M}, s, t)$ *and* $(\mathbf{M}', s', t')$ *satisfy the same propositional letters.*
Meet: $s = t$ *iff* $s' = t'$.
Zig$_{\mathsf{left}}$: *if there exists* $u \in W$ *such that* Rsu*, then there exists* $u' \in W'$ *such that* $R's'u'$ *and* $(\mathbf{M}, u, t) \underline{\leftrightarrow} (\mathbf{M}', u', t')$.
Zig$_{\mathsf{right}}$: *if there exists* $v \in W$ *such that* Rtv*, then there exists* $v' \in W'$ *such that* $R't'v'$ *and* $(\mathbf{M}, s, v) \underline{\leftrightarrow} (\mathbf{M}', s', v')$.
Zag$_{\mathsf{left}}$, Zag$_{\mathsf{right}}$: *those analogous clauses in the converse direction of* **Zig$_{\mathsf{left}}$** *and* **Zig$_{\mathsf{right}}$** *respectively.*[7]

With this definition, it is now easy to check that $(\mathbf{M}, w_1, w_2)$ and $(\mathbf{M}', v_1, v_2)$ in Figure 1 are bisimilar. Although the clauses above look rather routine, it is instructive to notice some subtle aspects of the definition that are in line with our previous observation: the condition **Atom** in effect just requires that $\mathsf{V}(s) \cap \mathsf{P_E} = \mathsf{V}'(s') \cap \mathsf{P_E}$ and $\mathsf{V}(t) \cap \mathsf{P_A} = \mathsf{V}'(t') \cap \mathsf{P_A}$, but s and s' may satisfy different properties p_A and p'_A, say, from $\mathsf{P_A}$, and t and t' may satisfy different properties p_E and p'_E, say, from $\mathsf{P_E}$. Moreover, the clause **Meet** aims to deal with the constant I, and the others are analogous to the zigzag conditions in standard situations.

By dropping the clause **Meet** above, we get the notion for LHS_{-I}, and by $(\mathbf{M}, s, t) \underline{\leftrightarrow}^- (\mathbf{M}', s', t')$ we denote the case that $(\mathbf{M}, s, t)$ and $(\mathbf{M}', s', t')$ are LHS_{-I}-bisimilar. With Definition 3, it holds that:

Proposition 3. *If* $(\mathbf{M}, s, t) \underline{\leftrightarrow} (\mathbf{M}', s', t')$*, then* $(\mathbf{M}, s, t)$ *and* $(\mathbf{M}', s', t')$ *satisfy the same* LHS*-formulas. Also, if* $(\mathbf{M}, s, t) \underline{\leftrightarrow}^- (\mathbf{M}', s', t')$*, then they satisfy the same* LHS_{-I}*-formulas.*

It can be proved by induction on the structure of LHS-formulas. Therefore, the language cannot distinguish between bisimilar models. However, our previous discussion indicates that having a very strong notion is never the final goal: it is equally important to ask whether the notion is also weak enough. This time we are going to present a positive result w.r.t. a class of models that are LHS-*saturated*:

Definition 4 (LHS-saturation). *A model* $\mathbf{M} = (W, R, V)$ *is said to be* LHS*-saturated, if for any set* Φ *of formulas and states* $w, v \in W$*, it holds that:*

- *If* Φ *is finitely satisfiable in* $R(w) \times v$*, then the whole set* Φ *is satisfiable in* $R(w) \times v$*, and*
- *If* Φ *is finitely satisfiable in* $w \times R(v)$*, then the whole set* Φ *is satisfiable in* $w \times R(v)$.

The notion is essentially obtained by adapting the so-called m-*saturation* [9] to fit into our logics. As usual, any finite model is LHS-saturated. Furthermore, in terms of infinite $\mathbf{M}$, it intuitively requires that $\mathbf{M}$ contains 'enough' states: for instance, if every finite subset of Φ can be satisfied by some pairs in $R(w) \times v$, then there must also be a pair satisfying Φ itself. By restricting Φ to the fragment without I, we have a notion for LHS_{-I}, called LHS_{-I}-*saturation*. Now we have enough background to show that:

[7] One may also like to treat LHS as a product logic over models containing two binary relations R_{left} and R_{right} on domain $W \times W$, and then explore expressive power or other properties of LHS with respect to the new setting. We leave a systematic study of relations between our logic and existing combined logics for future inquiry.

Proposition 4. *For all* $\mathbf{M}$ *and* $\mathbf{M}'$ *that are* LHS*-saturated, if* $(\mathbf{M}, s, t)$ *and* $(\mathbf{M}', s', t')$ *satisfy the same formulas of* LHS*, then it holds that* $(\mathbf{M}, s, t) \underline{\leftrightarrow} (\mathbf{M}', s', t')$*. Moreover, when* $\mathbf{M}$ *and* $\mathbf{M}'$ *are* LHS_{-I}*-saturated, if* $(\mathbf{M}, s, t)$ *and* $(\mathbf{M}', s', t')$ *satisfy the same formulas of* LHS_{-I}*, then it holds that* $(\mathbf{M}, s, t) \underline{\leftrightarrow}^- (\mathbf{M}', s', t')$*.*

It can be proved by showing that the modal equivalence relation itself is a bisimulation, but due to the page-limit constraints, details are omitted. Therefore, just as the usual case, w.r.t. the class of models that are $\mathsf{LHS}/\mathsf{LHS}_{-I}$-saturated, our notion of bisimulation coincides with the corresponding notion of modal equivalence.

Having shown that our novel notions behave well, we end this section with the following result concerning the relations among aforementioned varieties of bisimulations:

Proposition 5. *With respect to the three varieties of bisimulations* $\underline{\leftrightarrow}^s$*,* $\underline{\leftrightarrow}$ *and* $\underline{\leftrightarrow}^-$*, we have the following:*

(1). Both $\underline{\leftrightarrow}^s$ *and* $\underline{\leftrightarrow}$ *are strictly stronger than* $\underline{\leftrightarrow}^-$*:* $\underline{\leftrightarrow}^s$ *entails* $\underline{\leftrightarrow}^-$ *and* $\underline{\leftrightarrow}$ *entails* $\underline{\leftrightarrow}^-$*, but the converse directions do not hold.*
(2). $\underline{\leftrightarrow}^s$ *and* $\underline{\leftrightarrow}$ *are incomparable: they do not entail each other.*

Proof. We show the two claims one by one.

(1). The relation between $\underline{\leftrightarrow}^s$ and $\underline{\leftrightarrow}^-$ follows from Proposition 1, 2 and 4. Also, it is obvious that $\underline{\leftrightarrow}$ is stronger than $\underline{\leftrightarrow}^-$. For an example, consider the two models given in Figure 2: it holds $(\mathbf{M}, w_1, w_1) \underline{\leftrightarrow}^- (\mathbf{M}', v_1, v_1)$, but $\mathbf{M}, w_1, w_1 \models \langle \mathsf{left} \rangle \langle \mathsf{right} \rangle \neg I$ and $\mathbf{M}', v_1, v_1 \not\models \langle \mathsf{left} \rangle \langle \mathsf{right} \rangle \neg I$. Now, by Proposition 3, we do not have $(\mathbf{M}, w_1, w_1) \underline{\leftrightarrow} (\mathbf{M}', v_1, v_1)$.

(2). Consider the models in Figure 2. It is not hard to see that the states w_1 and v_1 cannot be distinguished by the basic modal language, but this would not be the case when we consider the logic LHS. Thus, standard bisimulations need not be bisimulations of LHS. On the other hand, using the models in Figure 1, it is not hard to see that bisimulations of LHS may also be excluded by the notion of standard bisimulation. This completes the proof. □

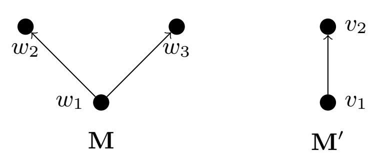

Fig. 2. $(\mathbf{M}, w_1, w_1) \underline{\leftrightarrow}^- (\mathbf{M}', v_1, v_1)$, but not $(\mathbf{M}, w_1, w_1) \underline{\leftrightarrow} (\mathbf{M}', v_1, v_1)$.

Properties of LHS- and LHS_{-I}- bisimulation explored here are very basic, and several further questions are worth studying. For instance,

Open problem. What is the computational complexity of checking for bisimulation of LHS or LHS_{-I}? Are they as complex as each other?

4 Towards undecidability of the satisfaction problem

Essentially, LHS introduces a propositional constant to deal with equality in a modal logic framework. This universally accepted relation of indiscernibility is simple in nature. However, as we mentioned in section 1, there are various elegant examples of logics that suggest that taking this relation into account may change previously decidable logics (without equality) into undecidable ones. In this section, we are going to contribute one more instance to this class: in what follows, we first show that LHS does not have the tree model property or the finite model property, and then prove that the satisfiability problem for LHS is undecidable.

Usually, the tree model property and the finite model property are positive signals for the computational behaviors of a logic (cf. e.g., [9]). However, in what follows, we will show that our logic LHS lacks both the properties. Let us begin with a simple result concerning the tree model property:

Proposition 6. *The logic* LHS *does not have the tree model property.*

Proof. Consider the following formula:

$$\varphi_r := I \wedge \langle \mathsf{left} \rangle \top \wedge [\mathsf{left}] I$$

It is easy to see that it is satisfiable. Also, let $\mathbf{M} = (W, R, \mathsf{V})$ and $u, v \in W$ such that $\mathbf{M}, u, v \vDash \varphi_r$. From I it follows that $u = v$. Also, the conjunct $\langle \mathsf{left} \rangle \top$ indicates that the state u has successors, i.e., $R(u) \neq \emptyset$. Moreover, for all $s \in R(u)$, we have $s = v$. Therefore, $R(u) = \{u\}$. Consequently, the model $\mathbf{M}$ cannot be a tree. The proof is completed. □

Moreover, by constructing a '*spy-point*' [10], i.e., all states that are reachable from u in n-steps can also be reached in one step, we can also establish the following:

Theorem 1. *The logic* LHS *lacks the finite model property.*

Proof. Let φ_∞ be the conjunction of the following formulas:

$$(F1) \qquad I \wedge [\mathsf{left}] \neg I$$
$$(F2) \qquad \langle \mathsf{left} \rangle [\mathsf{left}] \bot$$
$$(F3) \qquad [\mathsf{left}] \langle \mathsf{right} \rangle (\neg I \wedge \langle \mathsf{right} \rangle \top \wedge [\mathsf{right}] I)$$

Let us briefly comment on the intuition underlying these formulas. First, $(F1)$ shows that the two states in the current graph model are the same and the point is irreflexive. Also, formula $(F2)$ states that the point can reach a state that is a dead end having no successors. Additionally, the last formula, motivated by [20], indicates that the point has more than one successor and for all its successors i, there is also another successor j of i such that i has j as its only successor.

After presenting the basic ideas of those formulas, we show that the formula φ_∞ is satisfiable. Consider the model $\mathbf{M}_\infty = (W, R, \mathsf{V})$ that is defined as follows:

- $W := \{s\} \cup \mathbb{N}$

- $R := \{\langle s, i\rangle \mid i \in \mathbb{N}\} \cup \{\langle i+1, i\rangle \mid i \in \mathbb{N}\}$
- For all $p \in \mathsf{P_A} \cup \mathsf{P_E}$, $\mathsf{V}(p) := \emptyset$.

See Figure 3 for an illustration. By construction, it can be easily checked that the formula holds at (s, s), i.e., $\mathbf{M}_\infty, s, s \vDash \varphi_\infty$.

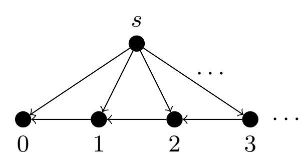

Fig. 3. The model $\mathbf{M}_\infty$.

Next, let $\mathbf{M} = (W, R, \mathsf{V})$ be an arbitrary model such that $u \in W$ and $\mathbf{M}, u, u \vDash \varphi_\infty$. We are going to show that W is infinite. To do so, we claim that the model contains the following sequence of states of $\mathbf{M}$:

$$w_0, w_1, w_2, w_3, w_4, \cdots$$

such that for all $i \in \mathbb{N}$, the following conditions hold:

P1. $\mathbf{M}, w_i, w_{i+1} \vDash \neg I \wedge \langle \mathsf{right}\rangle\top \wedge [\mathsf{right}]I$

P2. $\langle u, w_i\rangle \in R$

P3. $R(w_0) = \emptyset$, and for $1 \leq i$, $R(w_i) = \{w_{i-1}\}$

By making an induction on $i \in \mathbb{N}$, we show that there is always such a sequence of those w_i's.

First, let us consider the basic case that $i = 0$. As $\mathbf{M}, u, u \vDash (F2)$, we know that there is $w_0 \in W$ such that Ruw_0 and $\mathbf{M}, w_0, u \vDash [\mathsf{left}]\bot$. Therefore, $R(w_0) = \emptyset$, i.e., we have already obtained the dead end. Moreover, by formula $(F3)$, it holds $\mathbf{M}, w_0, u \vDash \langle \mathsf{right}\rangle(\neg I \wedge \langle \mathsf{right}\rangle\top \wedge [\mathsf{right}]I)$. Therefore, there exists $w_1 \in W$ such that Ruw_1, $w_0 \neq w_1$ and $R(w_1) = \{w_0\}$. Now, it is not hard to see that the clauses P1-P3 hold for both w_0 and w_1.

Now, suppose that we have already had all those states $w_{i \leq n}$, and we proceed to show that there exists w_{n+1} satisfying the conditions P1-P3. By the induction hypothesis, we have Ruw_n. Now, from the formula $(F3)$, it follows that $\mathbf{M}, w_n, u \vDash \langle \mathsf{right}\rangle(\neg I \wedge \langle \mathsf{right}\rangle\top \wedge [\mathsf{right}]I)$. So, there is a state $w_{n+1} \in W$ such that Ruw_{n+1} and $\mathbf{M}, w_n, w_{n+1} \vDash \neg I \wedge \langle \mathsf{right}\rangle\top \wedge [\mathsf{right}]I$. This indicates that w_{n+1} satisfies the requirements P1 and P2. Also, as $\neg I$, $w_n \neq w_{n+1}$. Furthermore, from $\mathbf{M}, w_n, w_{n+1} \vDash \langle \mathsf{right}\rangle\top \wedge [\mathsf{right}]I$, we know that $R(w_{n+1}) = \{w_n\}$, which indicates that the node satisfies P3 as well.

Moreover, by the property P3, we have $w_i \neq w_j$ whenever $i \neq j$. To be more specific, we have $\mathbf{M}, w_i, w_i \vDash \langle \mathsf{left}\rangle^i\top \wedge [\mathsf{left}]^{i+1}\bot$ for each i. Therefore, we have infinitely many states w_i. So, the model $\mathbf{M}$ is infinite. The proof is completed. □

4.1 Undecidability

Now, by encoding the $\mathbb{N} \times \mathbb{N}$ *tiling problem*, we show that the logic LHS is undecidable. A *tile* t is a 1×1 square, of the fixed orientation, with colored edges *right(t), left(t), up(t)* and *down(t)*. The $\mathbb{N} \times \mathbb{N}$ tiling problem is: given a finite set $T = \{t^1, \cdots, t^n\}$ of tile types, is there a function $f : \mathbb{N} \times \mathbb{N} \to T$ such that *right(f(n,m)) = left(f(n+1,m))* and *up(f(n,m)) = down(f(n,m+1))*? The problem is known to be undecidable (see [8]).

Theorem 2. *The satisfiability problem of logic* LHS *is undecidable.*

The proof is given by reduction of the tiling problem to the satisfaction problem of LHS, and is provided in the Appendix. It is worth noting that the proof essentially indicates the undecidability of the class of logics generalizing our framework to capture the games with $2 \leq n \in \mathbb{N}$ players.

A closely related problem is that of model-checking, and it is important to study the complexity of the model-checking problem for our logic. In contrast to the satisfaction problem, we believe that the model checking problem for LHS can be solved efficiently, that is, the problem lies in P.

Finally, it is not hard to see that logic LHS can be translated into first-order logic,[8] which then suggests that the logic itself is effectively axiomatizable. Consequently, a crucial direction is to explore the following:

Open problem. Is there a complete proof calculus for the logic LHS?

5 Related works

Graph games and modal logics. Motivated by a simple graph game of hide and seek, this paper belongs to a broader program [7] that promotes a study of graph game design in tandem with matching new modal logics. In recent years, several interesting new graph games have been studied. For instance, in sabotage games [6], a player moves along a link available to her on a graph to reach some fixed goal region, while her opponent removes an arbitrary link in each round to prevent her from reaching her goal. The games are captured by the sabotage modal logic [3], extending the basic modal logic with a link deletion operator. Further, games in which links are removed *locally* according to certain conditions which were expressed explicitly in the language have been studied in [21]. Moreover, several variants of sabotage games were applied to the learning/teaching scenarios [15], and their computational behaviors were analyzed. Following this direction, a new game setting allowing both link deletion and link addition was developed in [5] to capture some further features of the learning process. Closely related to the logic of [5], a class of *relation-changing logics*, containing operators to swap, delete or add links, was well explored in [2, 1]. Instead of modifying links, in poison games [13], a player can poison a node to make it unavailable to the opponent. These games have been studied with diverse modal approaches in [11] and [17]. Additionally, by updating valuation

[8] Although details of the translation are not described in the article, it is instructive to notice that unlike usual situations, LHS is not a fragment of the first-order logic with two free variables.

functions of models, a dynamic logic of local fact change was studied in [25], which captures a class of graph games in which properties of states might get affected by those of others.

Product logics with diagonal constant. Technically, our work is close to that of many-dimensional modal logics [22, 14]. In particular, a class of product logics was studied in [19, 20, 18] with the so-called *diagonal constant* δ.[9] In [20, 18] it was shown that $\mathbf{K}\times^{\delta}\mathbf{K}$, the product logic augmenting $\mathbf{K}\times\mathbf{K}$ with δ, lacks the finite model property and is undecidable, which seem very similar to our results at a first glance. However, our logic differs from those both conceptually and technically.

First, our formulas are evaluated at *pairs* of states, where each of the states can occur by itself (and, not just as a constituent of an ordered pair), which makes it possible for us to study the relationship between two states directly. In $\mathbf{K} \times^{\delta} \mathbf{K}$, even though formulas are evaluated at pairs of states, these pairs themselves form nodes in the domain. As a result, product logic cannot express the more fine-grained relation (i.e., identity) between the two components forming a pair. In [20, 18], δ is interpreted as a special subset of the domain, not necessarily consisting of pairs formed by the same components from those dimensions. Therefore, we can say that constant I explored in this article works at a meta level. In contrast, δ in [20, 18] is an object level notion.[10]

Next, techniques adopted to establish the undecidability of LHS are very different. Similar to all other product logics, various relations representing transitions of states in different dimensions are considered in [20, 18]. Moreover, the product nature endows the relations with possible interactions: say, *commutativity* and *confluence*. With such interactions, product logics obtain grid-like structures automatically. However, as illustrated in our proofs, a crucial step in proving undecidability of LHS was exactly to build such a shape. In other words, these extra efforts make our proof technically non-trivial.

6 Conclusion and future work

Summary. Motivated by the meeting/avoiding game, this paper studies a modal logic LHS that allows us to talk about moves for each player, as well as the situation of meeting. More specifically, formulas in this logic are evaluated at two states of the domain, representing positions of different players. A constant I expressing the meeting of two players is explored in depth, which adds a natural and novel treatment of equality in modal logics. We establish a series of results concerning its expressive power and computational behavior. A new notion of bisimulation for LHS is proposed, and is compared systematically with those of related logics. Further, we have proved that the logic does not enjoy the tree model property or the finite model property, and that the satisfiability

[9] For instance, in two dimensional models δ holds at a state (s, t) just in the case that $s = t$.

[10] But this does not exclude possible 'mixture' of the two lines of the frameworks: on one hand, technically LHS can be reduced to product logics with δ, and on the other hand, product models themselves can also be viewed as special models (with two relations) for LHS (and then I denotes the identity of *two* pairs).

problem of the logic is undecidable, which refutes a conjecture made by van Benthem and Liu in their recent paper [7].

Further directions. We mention a few directions that we would like to pursue immediately. Though we have obtained some basic results about LHS, more properties of the logic need to be explored. Several open problems have been formulated along the way, including the axiomatization of LHS, and issues regarding its expressive power. Regarding the language, the constant I seems rather simple and innocent, but surprisingly, our logic turned out to be undecidable. It makes sense to understand this phenomenon better, and possibly by investigating some alternative logics (e.g., the logic mentioned in Remark 1). In section 5, we have seen the differences between our work and product logics, and a systematic comparison is needed. As stated earlier, we have taken a high-level modeller's perspective to study the hide and seek game in this paper. We reason about players' observations and moves with the assumption that the whole graph and the players' positions at each stage of the game are available to us. For the next step, we will pursue strategic reasoning from the players' perspectives *in* the game. We will focus on technical issues like the epistemic aspects of the players and extend the current language with epistemic modalities to deal with those concepts.

Finally, as mentioned in various places, our work has a natural connection with the game of cops and robber in the vast literature of graph games (see, e.g., [12, 23]). We are exploring richer versions of these games, focusing on different characterization results of *cop-win graphs*. We have extended LHS with modal substitution operators [26] which enable us to express winning positions of players in the general sense, as discussed in section 2. We have also obtained some new results regarding cop-win characterizations. We will continue this line of research in the future.

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Appendix: Proof of Theorem 2

Proof. Let $T = \{T^1, \cdots, T^n\}$ be a finite set of tile types. For each $T^i \in T$ we use $u(T^i), d(T^i), l(T^i), r(T^i)$ to represent the colors of its up, down, left and right edges, respectively. We are going to define a formula φ_{tile} such that:

$$\varphi_{\mathsf{tile}} \text{ is satisfiable iff } T \text{ tiles } \mathbb{N} \times \mathbb{N}.$$

To do so, we will use three relations in models (W, R^s, R^r, R^u) in the proof to follow. In line with this, syntactically we have six operators $[\mathsf{left}]^\star$ and $[\mathsf{right}]^\star$ for $\star \in \{s, r, u\}$. Intuitively, all the relations describe the transitions of the left evaluation point and the right evaluation point of a graph model: in what follows, we are going to construct a spy point over relation R^s, and the relations R^u and R^r represent moving up and to the right, respectively, from the corresponding tile to the other.

These three relations are useful to present the underlying intuitions of the formulas that will be constructed. Also, they are helpful in making these formulas short and readable, facilitating a better understanding of the same. Crucially, this does not change the computational behavior of the original LHS: the three relations can be reduced to one relation as that of our standard models. For instance, given that we have three relations and the evaluation gap, we can mimic the three relations with a singular relation and 3×2 fresh propositional letters encoding, e.g., $[\mathsf{left}]^\star$ and $\langle\mathsf{right}\rangle^\star$ as $[\mathsf{left}](p^\star_E \to \cdots)$ and $\langle\mathsf{right}\rangle(p^\star_A \wedge \cdots)$, respectively. Definitely, to preserve the structure of those relations and truth of formulas, we need to be careful when defining the new relation and the valuation function. However, due to page-limit constraints, we forego those details here. Now, we proceed to present the details of φ_{tile}, whose components will be divided into four groups. Let us begin with the first one.

Group 1: Infinite many states induced by R^s and their 'scope'

$$
\begin{array}{ll}
(U1) & I \wedge [\mathsf{left}]^s \neg I \\
(U2) & \langle\mathsf{left}\rangle^s [\mathsf{left}]^s \bot \\
(U3) & [\mathsf{right}]^s \langle\mathsf{left}\rangle^s (\neg I \wedge \langle\mathsf{left}\rangle^s \top \wedge [\mathsf{left}]^s I) \\
(U4) & [\mathsf{left}]^s [\mathsf{right}]^s ([\mathsf{left}]^s \bot \wedge [\mathsf{right}]^s \bot \to I) \\
(U5) & [\mathsf{left}]^s [\mathsf{right}]^s (\langle\mathsf{left}\rangle^s \langle\mathsf{right}\rangle^s I \to I)
\end{array}
$$

Notice that formulas $(U1)$-$(U3)$ are just the R^s-version of the formulas in Theorem 1 that are used to create infinite models. Immediately, there exists an infinite sequence of states as follows:

$$w_0, w_1, w_2, \cdots$$

such that $R^s(w_{i+1}) = \{w_i\}$ and $R^s(w_0) = \emptyset$. Also, for the current evaluation pair (e.g., (s, s)), we have $\{w_i \mid i \in \mathbb{N}\} \subseteq R^s(s)$.

Now let us spell out what $(U4)$ and $(U5)$ express. Essentially, both the formulas establish a 'border' for the scope of nodes that are (directly or indirectly) reachable from s via relation R^s. Specifically, the formula $(U4)$ shows that $R^s(s)$ contains only a dead end which is exactly w_0 listed above, and moreover, the formula $(U5)$ indicates that for any $w_i, w_j \in R^s(s)$, if they can reach the same state, then we have $w_i = w_j$. See Figure 4 for two counterexamples

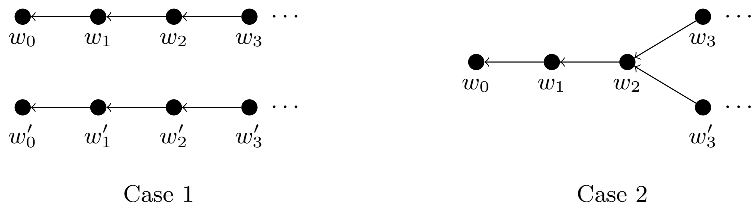

Fig. 4. Two impossible cases of the R^s-structure $R^s(s)$: Case 1 cannot satisfy $(U4)$, while Case 2 cannot satisfy $(U5)$.

without the properties of $(U4)$ or $(U5)$. From the two formulas, we know that $R^s(s) = \{w_i \mid i \in \mathbb{N}\}$.

Intuitively, we will use these $w_i's$ to represent tiles. To make this precise, beyond the simple linear order of R^s among those states, we still need to structure them with R^r and R^u in a subtler way. Our next group of formulas concerns some basic features of the two relations:

Group 2: Basic features of R^u and R^r

$(U6)$	$[\mathsf{left}]^s[\mathsf{left}]^\dagger\langle\mathsf{right}\rangle^s I$	$\dagger \in \{u, r\}$
$(U7)$	$[\mathsf{left}]^s(\langle\mathsf{left}\rangle^u\top \wedge \langle\mathsf{left}\rangle^r\top)$	
$(U8)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^u\neg I \wedge [\mathsf{left}]^r\neg I)$	
$(U9)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^\dagger\neg\langle\mathsf{left}\rangle^\dagger I)$	$\dagger \in \{u, r\}$

Before listing more formulas, let us briefly comment on these properties.

For all $i \in \mathbb{N}$, the formulas of $(U6)$ essentially give $R^r(w_i)$ and $R^u(w_i)$ a 'scope'. Specifically, they guarantee that $R^r(w_i), R^u(w_i) \subseteq \{w_0, w_1, \cdots\}$. Therefore, when considering the two relations, we only need to consider those $w_i's$, and there do not exist other states that are involved.

The formula $(U7)$ states that every w_i has successors via R^u and R^r, i.e., $R^u(w_i) \neq \emptyset$ and $R^r(w_i) \neq \emptyset$. Intuitively, this expresses that every tile has at least one tile above it and at least one tile to its right.

Also, the formula $(U8)$ indicates that for all $i \in \mathbb{N}$, we do not have $R^r w_i w_i$ or $R^u w_i w_i$. Moreover, formulas in $(U9)$ show that both the relations R^r and R^u are asymmetric.

Except those basic features captured by formulas of Group 2, what might be more important is our next group of formulas, which structure the states in a *grid* with R^r and R^u:

Group 3: Grid formed by R^u and R^r

$(U10)$	$[\mathsf{left}]^s[\mathsf{right}]^s(\langle\mathsf{left}\rangle^\dagger I \to [\mathsf{left}]^\dagger I)$	$\dagger \in \{u, r\}$
$(U11)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^u[\mathsf{right}]^r\neg I)$	
$(U12)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^u[\mathsf{left}]^r\neg I \wedge [\mathsf{left}]^r[\mathsf{left}]^u\neg I)$	
$(U13)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^u[\mathsf{left}]^r[\mathsf{left}]^u\neg I)$	
$(U14)$	$[\mathsf{left}]^s[\mathsf{right}]^s(I \to [\mathsf{left}]^u[\mathsf{right}]^r\langle\mathsf{left}\rangle^r\langle\mathsf{right}\rangle^u I)$	

Whereas $(U7)$ tells us that all the $w_i's$ have R^r- and R^u-successors, formulas in $(U10)$ state that every w_i has at most one R^r-successor and at most one

R^u-successor. Thus, both $(U7)$ and $(U10)$ ensure that the transitions between those $w_i's$ via R^u and R^r are essentially *functions*: precisely, for all $i \in \mathbb{N}$ and $\dagger \in \{u, r\}$, $R^\dagger(w_i)$ is a singleton.

Moreover, formula $(U11)$ suggests that the R^r-successor and the R^u-successor of a tile are different: for all $i \in \mathbb{N}$, $R^r(w_i) \cap R^u(w_i) = \emptyset$. That is, a tile cannot be above as well as to the right of another tile.

Additionally, $(U12)$ shows that no tile can be both above/below and to the right/left of another tile, and $(U13)$ disallows cycles following successive steps of the R^u, R^r and R^u relations, in this order. Formula $(U14)$ states the property of '*confluence*': for all tiles w_i, w_j, w_k, if $R^u w_i w_j$ and $R^r w_i w_k$ hold, then there exists another tile w_n such that $R^r w_j w_n$ and $R^u w_k w_n$ hold. Now, the tiles are arranged in a grid.

Now, it remains to set a genuine tiling, which can be achieved by our fourth group of formulas. Very roughly, in usual cases this work is often routine when we have an infinite grid-like model (cf. [9]). Let us present the details here:

Group 4: Tiling the model

$$(U15) \qquad [\mathsf{left}]^s(\bigvee_{1\leq i\leq n} t_E^i \wedge \bigwedge_{1\leq i<j\leq n} \neg(t_E^i \wedge t_E^j))$$

$$(U16) \qquad [\mathsf{right}]^s(\bigvee_{1\leq i\leq n} t_A^i \wedge \bigwedge_{1\leq i<j\leq n} \neg(t_A^i \wedge t_A^j))$$

$$(U17) \qquad [\mathsf{left}]^s[\mathsf{right}]^s(I \to \bigvee_{1\leq i\leq n} (t_E^i \wedge t_A^i))$$

$$(U18) \qquad [\mathsf{left}]^s(\bigwedge_{1\leq i\leq n} (t_E^i \to \langle\mathsf{left}\rangle^u \bigvee_{1\leq j\leq n, u(T_i)=d(T_j)} t_E^j))$$

$$(U19) \qquad [\mathsf{left}]^s(\bigwedge_{1\leq i\leq n} (t_E^i \to \langle\mathsf{left}\rangle^r \bigvee_{1\leq j\leq n, r(T_i)=l(T_j)} t_E^j))$$

Formulas $(U15)$-$(U16)$ indicate that a node can be occupied 'two' tiles t_E^i and t_A^j. As one node can only be occupied by exactly one tile, the statement here may look a bit strange. However, we would like to argue that essentially there exists no problem, see our discussion on formula $(U17)$ below.

By formula $(U17)$, for every fixed i, when both t_E^i and t_A^j hold at a node, then we have $i = j$, i.e., they are of the same type T^i. In this sense, we can say that 'E' and 'A' are just 'position-labels' to refer to the evaluation nodes in the current graph model, and a node in the model is essentially occupied by exactly one tile. Moreover, for the same reason, although for each T^i, we have different propositional atoms t_A^i and t_E^i, all types of tiles we use are exactly those given by the original T, but not any extra ones.

Finally, the ideas of formulas $(U18)$ and $(U19)$ are routine: the former one states that colors match going up, while the latter expresses that they match going right.

Now, let φ_{tile} be the conjunctions of all formulas listed in the four groups. Based on our analyses above, any model satisfying φ_{tile} is a tiling of $\mathbb{N} \times \mathbb{N}$.

On the other hand, we still need to show the other direction. Now suppose that a function $f : \mathbb{N} \times \mathbb{N} \to T$ is a tiling of $\mathbb{N} \times \mathbb{N}$. Define a model $\mathbf{M}_t = (W, R^s, R^u, R^r, \mathsf{V})$ in the following:

- $W := \{s\} \cup (\mathbb{N} \times \mathbb{N})$
- R^s consists of the following:
  - For all $x \in \mathbb{N} \times \mathbb{N}$, $\langle s, x \rangle \in R^s$
  - For all $\langle n, 0 \rangle \in \mathbb{N} \times \mathbb{N}$ with $1 \leq n$, $\langle \langle n+1, 0 \rangle, \langle 0, n \rangle \rangle \in R^s$
  - For all other $\langle n, m+1 \rangle \in \mathbb{N} \times \mathbb{N}$, $\langle \langle n, m+1 \rangle, \langle n+1, m \rangle \rangle \in R^s$
- $R^u := \{\langle \langle n, m \rangle, \langle n, m+1 \rangle \rangle \mid n, m \in \mathbb{N}\}$
- $R^r := \{\langle \langle n, m \rangle, \langle n+1, m \rangle \rangle \mid n, m \in \mathbb{N}\}$
- $\mathsf{V}(t^i_E) = \mathsf{V}(t^i_A) = \{\langle n, m \rangle \in \mathbb{N} \times \mathbb{N} \mid f(\langle n, m \rangle) = T^i\}$, for all $i \in \{1, \cdots, n\}$
- $\mathsf{V}(p_E) = \mathsf{V}(q_A) = \emptyset$, for all other $p_E, q_A \in \mathsf{P_E} \cup \mathsf{P_A}$.

Figure 5 presents a crucial fragment of the model. By construction, it is not hard to check that $\mathbf{M}_t, s, s \vDash \varphi_{\mathsf{tile}}$. This completes the proof. □

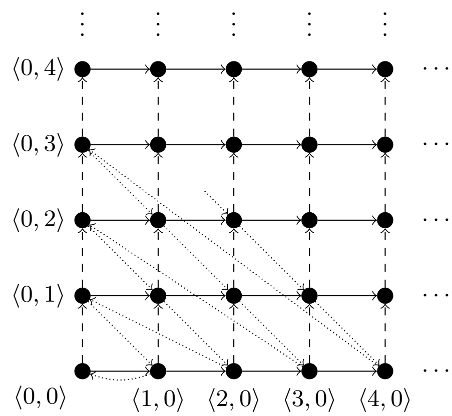

Fig. 5. The restriction of the structure of $\mathbf{M}_t$ to $\mathbb{N} \times \mathbb{N}$, where the resulting R^s, R^u, R^r are represented by dotted-, dashed- and solid-arrows respectively. To obtain the whole structure, we just need to add the state s and draw a dotted-arrow from s to each of the members of $\mathbb{N} \times \mathbb{N}$.