Rely more or less, for better or for worse: Intertwining reliability and preferences

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Abstract

Dealing with preferences is a ubiquitous part of our daily lives, and thus, preference forms an integral part of our daily decision making processes. If we consider preferences among groups of agents, deliberation often leads to changes in preferences of an agent, influenced by the opinions of others, depending on how reliable these agents are according to the agent under consideration. Sometimes, it also leads to changes in the opposite direction, that is, reliability over agents gets updated depending on their preferences. There are various formal studies of preference change based on reliability and/or trust, but not the other way around − this work contributes to the formal study of the latter aspect, that is, on reliability change based on agent preferences. In process, some policies of preference change based on agent reliabilities are also discussed. We introduce a novel formal study of the relevant changes in agent reliabilities in decision making scenarios which is an integral part of artificial intelligence studies in the current day. A two-dimensional hybrid language is proposed to describe such processes, and axiomatizations and decidability are discussed.

1 Introduction

Deliberation forms an important component in any decision-making process. It is basically a conversation through which individuals provide their opinions regarding certain issues, give preferences among possible choices, justify these preferences. This process may lead to changes in their opinions, because they are influenced by one another. A factor that sometimes plays a big role in enforcing such changes, is the amount of reliability the agents have on one another's opinions. Such reliabilities may change as well through this process of deliberation, e.g. on hearing someone else's preferences about a certain issue, one can start or stop relying on that person's opinion. One may tend to unfriend certain friends hearing about their preferences regarding certain issues (e.g., see [7]).

Formal studies on preferences (cf. [2, 11]; see [10] for an overview) and trust (cf. [23, 9, 21]) are abound in the literature on logic in artificial intelligence. Recently, there has been work on relating the notions of belief and trust, e.g., about agents changing their beliefs based on another agent's announcement depending on how trustworthy that agent is about the issue in question (e.g., see [25]). And, also on relating preference and reliability, e.g., about agents changing their preferences based on another agent's preferences, on whom he or she relies the most [16, 17, 33]. A pertinent issue that arises in this context is: an agent's assessment of another individual's reliability might change as well, based on the other agent's preferences. How does one model such changes in reliability? This work precisely provides a way to answer this question in an attempt to broaden the scope of formal modelling in artificial intelligence. We focus on reliability changes based on (public) announcement of individual preferences and we provide formal frameworks to describe such changes. In process, we discuss some policies of preference change as well, which fit well with the way we model reliability of agents over other agents. Note that the notion

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of reliability considered here is not topic-based (in contrast to the notion of trust described in [25]) but deals with only comparative judgements about agents (cf. Section 2 for details). To give an idea of the interdependence of reliability and preferences in terms of their dynamics consider Figure 1.

$$\begin{tikzcd}
F = (W, (R_a)_{a\in A}) & F = (W, (R_a)_{a\in A}) \arrow[r, mapsto, "f_{G'}"] & F' = (W, (R'_a)_{a\in A}) & F' = (W, (R'_a)_{a\in A}) \\
G = (A, (S_a)_{a\in A}) \arrow[r, mapsto, "g_F"] & G' = (A, (S'_a)_{a\in A}) & G' = (A, (S'_a)_{a\in A}) \arrow[r, mapsto, "g_{F'}"] & G'' = (A, (S''_a)_{a\in A})
\end{tikzcd}$$

Figure 1: Preference ordering and reliability ordering influencing changes in each other

A preference ordering over worlds (F) can induce a change in the reliability ordering over agents (G) to form a new reliability ordering (G'). On the other hand, a reliability ordering over agents (G') can induce a change in the preference ordering over worlds (F) to form a new preference ordering (F'). And, we can have iteration of such changes, sometimes leading to a consensus in the preferences in certain cases, where, just a sequence of preference changes might lead us to oscillating results.

This is the basic idea of reliability and preference changes that we will discuss in this work. As discussed above, the novelty of this work lies in the consideration of reliability changes based on preference orderings. Now, the question is how do we define such reliability changes based on preference orderings? One possible way could have been to consider various notions of distances between different preference orderings (cf. [6]). However, for the sake of simplicity and to maintain the deliberative intuition, we use the idea of preference matches between the agents, that is, on how many states do the preferences of two agents match. A motivation to use such matches is the fact that in our deliberation process, we do not look for distance measures to describe similarities/differences between preferences, but some simple notion like preference matching to get to our decisions. Further details on this can be found in Section 3.

To model preference orderings among worlds and reliability ordering among agents, we introduce a two-dimensional hybrid logic framework extending the basic logic proposed in [16, 17, 33] in the line of those developed in [14, 28, 29]. We add dynamic operators to model reliability and preference changes. The main novelty of this work is that reliability changing policies based on agent preferences are introduced and studied formally, which has not been dealt with before (to the best of the knowledge of the authors). In addition, reliabilities are modelled (more naturally) as pre-orders (instead of total orders [16, 17]), and preference changing policies are modified accordingly [33]. We also consider reliabilities as total pre-orders and revise the preference change operators proposed in [16] accordingly. The proposed logic is expressive enough to deal with both these kinds of changes. The main ingredients of our work are:

- a two-dimensional hybrid language to discuss about preferences and reliabilities of agents;
- a sound and complete axiomatization of the general version of the proposed hybrid logic and decidability of a finite variant of the hybrid framework which is enough for modelling relevant situations;
- reliability change operators based on preferences of agents together with a generalized version of such operators;
- modelling reliability dynamics in the proposed hybrid framework, together with a sound and complete axiomatization of the dynamics;
- preference change operators based on reliabilities of agents;
- modelling preference dynamics in the proposed hybrid framework, together with a sound and complete axiomatization of the dynamics.

There is an inherent relationship between belief and trust which govern our day-to-day life which might bring out the following question: Why the reliability change operators are not defined in terms of agents' beliefs, rather than agents' preferences? We argue that in many decision making situations, e.g., in our running example that we introduce below, we deliberate and make decisions 'on the fly' based on the evolving situations after getting to know each others' preferences, without going into the deeper question: How much do I believe in him/her? This is exactly where our proposed framework positions itself. As mentioned above, belief of an agent also plays an important role in the dynamics of her trust/reliability and we leave that study for a future occasion.

In what follows, we discuss in detail the basics of the two-dimensional hybrid logic framework in Section 2, which is used to model reliability and preference dynamics in the later part. In Section 3, we introduce several policies of reliability change and express them in our proposed logic. In Section 4, we discuss several preference change policies based on agent reliabilities which are existing in the current literature and express them in our proposed logic. Section 5 concludes our paper.

This article is an extended version of [15]. The main differences are: the reliability ordering in the preference-relibility frame is assumed to be a pre-order, the syntax of the two-dimensional hybrid logic is both simplified and refined, and detailed soundness and completeness proofs of the two-dimensional hybrid logic expressing agent preferences and reliabities are provided (see Section 2 and Appendix); a general reliability upgrade operator is proposed together with a completeness result in the logical framework (see Section 3); a general lexicographic preference upgrade operator based on total pre-order reliabilities is proposed together with a completeness result in the logical framework (see Section 4); a preference upgrade operator corresponding to pre-order reliabilities is added from [33] and a completeness result is provided in the logical framework (see Section 4); the running example is reviewed throughout so as to consider both total and non-total pre-orders on reliabilities of agents; and finally, consideration of beliefs is left for a future occasion - a conditional goal operator is introduced to describe agents' preferences (see Section 2).

2 Preference, reliability and a two-dimensional hybrid logic

Let us first motivate our assumptions on preference and reliability orders that we make below in the lines of [16, 33]. As mentioned earlier, we are modelling situations akin to joint deliberation where agents announce their preferences. Each agent can change her preferences upon getting information about the other agents' preferences, influenced by her reliability over agents. Agents can also change their opinions regarding how reliable they think the other agents are in comparison to themselves, influenced by the announced preferences of those agents.

The agents' preferences are represented by binary relations (as in [2, 19] and further references therein), which is typically assumed to be reflexive and transitive. This paper also follows this assumption, and we note that this assumption do allow the possibility of incomparable worlds.

The notion of *reliability* is related to that of *trust*, a well-studied concept (e.g., in [12]), with several proposals for its formal representation, e.g. an attitude of an agent who believes that another agent has a given property [13]. One also says that "an agent i trusts agent j's judgement about φ" (called "trust on credibility" in [8]). Trust can also be defined in terms of other attitudes, such as knowledge, beliefs, intentions and goals (e.g., [8, 21]), or as a semantic primitive, typically by means of a *neighbourhood function* [23]. Some others (e.g., [25]) deal with *graded trust*.

Reliability as discussed here is closer to the notion of trust in [22], where it is understood as an ordering among sets of sources of information (cf. the discussion in [18]). Such a notion of reliability does not yield any *absolute* judgements ("i relies on j's judgement [about φ]"), but only *comparative* ones ("for i, agent k is at least as reliable as agent j"). For the purposes of this work, similar to [16, 33], such comparative judgements suffice.

In contrast to [16, 15], our reliability relation is assumed to be a reflexive and transitive relation (similar to [33]), i.e., a preorder. Reflexivity and transitivity are, more often than not, natural requirements for an ordering. [1]

The changes in reliability for an agent depend on the information assimilated (similar to approaches like [27]), in particular, about the other agents' preferences. The focus of this work is on the outcomes of a joint deliberation, so let A be a *finite non-empty* set of agents (we often assume that $|A| = n \geq 2$).

Definition 1. A *PR* (*preference/reliability*) *frame* F is a tuple $(W, \{\leq_i, \preccurlyeq_i\}_{i\in A})$ where **(1)** W is a finite non-empty set of worlds; **(2)** $\leq_i \subseteq W \times W$ is a preorder (i.e., a reflexive and transitive relation), agent i's *preference relation* among worlds in W ($u \leq_i v$ is read as "world v is at least as preferable as world u for agent i"); **(3)** $\preccurlyeq_i \subseteq A \times A$ is a pre-order, agent i's *reliability relation* among agents in A ($j \preccurlyeq_i k$ is read as "agent k is at least as reliable as agent j for agent i").

We define the following abbreviations.

- $u <_i v$ ("u is less preferred than v for agent i") means $u \leq_i v$ and $v \not\leq_i u$,
- $u \simeq_i v$ ("u and v are equally preferred for agent i") means $u \leq_i v$ and $v \leq_i u$,
- $j \prec_i k$ ("j is less reliable than k for agent i") is defined as $j \preccurlyeq_i k$ and $k \not\preccurlyeq_i j$,
- $j \approx_i k$ ("j and k are equally reliable for agent i") means $j \preccurlyeq_i k$ and $k \preccurlyeq_i j$.

Example 1. The following provides an apt example of the situations we would like to model:

> *Our Running Example*: Consider three flatmates Isabella, John and Ken discussing about redecorating their house and they were wondering whether to put a print of Monet's picture on the left wall or on the right wall of the living room. Isabella and Ken prefer to put it on the right wall, while John wants to put it on the left. Isabella has more faith in John's taste than on hers, John has more faith in Isabella's taste than on his and Ken has full faith in his own taste in comparison to the others' taste. But it so happens that on hearing about John's and Ken's choices, Isabella starts relying more on Ken than on John, whereas even after hearing about Isabella's and Ken's choices John's reliability attribution to Isabella does not change. Thus, Isabella's and Ken's choices remain the same after considering the change of reliability considerations in Isabella. Moreover, John's choice changes being influenced by Isabella's. Thus, they come to an agreement.
>
> Note that if we did not take the reliability change under consideration, as long as Isabella's and John's preferences are different and each think that the other's taste is better (by taking their preferences into consideration), the three flatmates would never reach an agreement.

Put $A = \{i, j, k\}$, where i, j, and k represent Isabella, John, and Ken, respectively. By denoting with w_x the world where Monet's picture is at wall x ($x = l, r$), the example's situation can be represented by a *PR* frame $F^{ex} = (\{w_l, w_r\}, \{\leq_y, \preccurlyeq_y\}_{y\in A})$ in which the preference orders are given by: $w_l <_i w_r$, $w_r <_j w_l$ and $w_l <_k w_r$, and the reliability orders should satisfy at least: $i \prec_i j$; $j \prec_j i$; $i \prec_k k$, $j \prec_k k$. It is noted that the reliability orders in F^{ex} have some degree of freedom which comes from our interpretation of how we include Ken in both Isabella's and John's reliability considerations. One way of interpreting this is to define the reliability ordering as: $k \prec_i i \prec_i j$; $k \prec_j j \prec_j i$; $i \prec_k k$, $j \prec_k k$. We write this frame as F^{ex}_1. Another way is to stipulate that k is not connected to i and j in terms of both i's and j's reliability. We write this frame as F^{ex}_2. We note that Ken's reliability ordering is the same in both the cases. The reader may refer to [15] for the case where reliability orderings of Isabella and John form total pre-orders.

[1] The reader may wonder why the reliability relation is transitive. This is because our reliability relation is comparative, and so, it is indexed by a given agent. Once we fix such an agent, we can relate agents (possibly including the agent) in terms of reliability from the fixed agent's viewpoint. Then, it is natural to require transitivity for such an indexed reliability relation. In this sense, it is also remarked that our indexed reliability relation is different from an absolute notion of "trust', because this notion may not be transitive, i.e., a may not trust c, even if a trusts b and b trusts c.

Agents	Preference Ordering	Reliability Ordering (1)	Reliability Ordering (2)
Isabella (I)	$\mathit{left} \longrightarrow \mathit{right}$	$\mathcal{K} \longrightarrow \mathcal{I} \longrightarrow \mathcal{J}$	$\mathcal{I} \longrightarrow \mathcal{J}$ $\mathcal{K}$
John (J)	$\mathit{right} \longrightarrow \mathit{left}$	$\mathcal{K} \longrightarrow \mathcal{J} \longrightarrow \mathcal{I}$	$\mathcal{J} \longrightarrow \mathcal{I}$ $\mathcal{K}$
Ken (K)	$\mathit{left} \longrightarrow \mathit{right}$	$\mathcal{I} \longrightarrow \mathcal{K}$ $\mathcal{J} \longrightarrow \mathcal{K}$	$\mathcal{I} \longrightarrow \mathcal{K}$ $\mathcal{J} \longrightarrow \mathcal{K}$

Figure 2: Tabular representations of F_1^{ex} and F_2^{ex} with the corresponding preference and reliability orderings are given by their Hasse diagrams.

In [16], Ghosh and Velázquez-Quesada proposed a language to talk about the preference changes and their effects based on public announcements of individual preferences, and subsequently variants and extensions of that work in terms of preference change can be found in [17, 33] . The main goal of this paper is to extend the work done in [16] to express both reliability as well as preference changes and their effects on each other. To this end, following the semantic idea of [29], we extend the syntax of [16] for the static language into a two-dimensional syntax with the help of dependent product of two hybrid logics [28].

The reader may wonder why we introduce two-dimensional hybrid logics in this paper. There are essentially two reasons. The first reason is that we need to talk about agents and their reliability structure explicitly in our syntax. This requires us to have syntactic names for agents in the syntax. Hybrid formalism allows us to have such syntactic names in terms of a new sort of propositional variables called *agent-nominals*. The second reason consists in considering a rich variety of preference and reliability changing operations. To cover such a variety of dynamic operators, our syntax should have enough program constructions, because we describe preference and reliability changing operations in terms of compounded programs. Moreover, to axiomatize logics containing a program construction of the intersection of two programs, both world- and agent-nominals play an indispensable role in our axiomatization. The use of nominals to define the intersection of programs can be traced back to [14] (cf. [1, pp.28–31]).

2.1 Syntax and semantics

Let P be a countable infinite set of propositional variables, let $\mathsf{N}_1 = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \dots\}$ be a countable infinite set of world-nominals (syntactic names for worlds) and let $\mathsf{N}_2 = \{\mathsf{i}, \mathsf{j}, \mathsf{k}, \dots\}$ be a countable infinite set of agent-nominals (syntactic names for agents). Our intuitive explanations of the syntactic vocabularies follow the definition of formulas and programs.

Definition 2 (Language $\mathcal{HL}$)**.** *Formulas* $\varphi, \psi, \dots$ and *relational expressions* (or *program terms*) $\pi, \rho, \dots$ of the language $\mathcal{HL}$ are given by

$$
\begin{aligned}
\varphi, \psi &::= p \,|\, \mathsf{a} \,|\, \mathsf{i} \,|\, \neg\varphi \,|\, \varphi \vee \psi \,|\, @_{\mathsf{i}}\varphi \,|\, @_{\mathsf{a}}\varphi \,|\, \langle\pi\rangle\varphi, \\
\pi, \rho &::= \leq \,|\, 1_W \,|\, \sqsubseteq_{\mathsf{k}} \,|\, 1_A \,|\, \pi^{-1} \,|\, -\pi \,|\, \pi \cup \rho \,|\, \pi \cap \rho \,|\, \pi \upharpoonright \mathsf{j} \,|\, ?(\varphi, \psi),
\end{aligned}
$$

where $p \in \mathsf{P}$, $\mathsf{a} \in \mathsf{N}_1$ and $\mathsf{i}, \mathsf{j} \in \mathsf{N}_2$. Propositional constants $(\top, \bot)$, other Boolean connectives $(\wedge, \rightarrow, \leftrightarrow)$ and the dual modal universal operators $[\pi]$ are defined as usual, e.g. $[\pi]\,\varphi := \neg\langle\pi\rangle\,\neg\varphi$.

Operators of the form $@_\mathsf{n}$ and $@_\mathsf{i}$ are called *satisfaction operators* in hybrid logics. A formula $@_\mathsf{a}\varphi$ is read as “the current agent satisfies φ in the world named by a”and $@_\mathsf{i}\varphi$ as “agent i satisfies φ in the current world.” Without satisfaction operators, a formula φ is read as “the current agent satisfies the property φ in the current world” or indexically as “I am φ in the current world”. The set of relational expressions contains the preference and reliability relations ($\leq$, $\sqsubseteq_\mathsf{k}$), their respective universal relations (1_W, 1_A) and an additional construct of the forms $\pi \upharpoonright \mathsf{j}$ (needed for defining distributed preference later in Section 4) and $?(\varphi,\psi)$ (a generalization of the test operator in [20], also explained below), and it is closed under complement, converse, union and intersection operations over relations.

The formulas are interpreted in terms of *world-agent* pairs below, and we may read $[\leq]\varphi$ as “in all worlds which the current agent considers at least as good as the current world, the current agent satisfies φ”. Then $@_\mathsf{k}@_\mathsf{a}\langle\leq\rangle\mathsf{b}$ can be read as “for agent k, world b is at least as preferable as world a.” Moreover, we may read $\langle\sqsubseteq_\mathsf{k}\rangle\varphi$ as “from agent k’s perspective, there is a more or equally reliable agent j than the current agent such that j satisfies φ.” For example, $@_\mathsf{i}\langle\sqsubseteq_\mathsf{k}\rangle\mathsf{j}$ can be read as “from agent k’s perspective, agent j is more or equally reliable than agent i.” A formula $\langle\sqsupseteq_\mathsf{k}\rangle\varphi$ is read as “from agent k’s perspective, there is a less or equally reliable agent j than the current agent such that j satisfies φ.” To encode the information of *PR*-frames, these examples explain why the atomic program $\leq$ is not indexed by agent-nominals but the atomic program $\sqsubseteq_\mathsf{k}$ is indexed by an agent-nominal.

We note that the program construction $?(\varphi,\psi)$ (check if the first element of a given pair of worlds satisfies φ and if the second satisfies ψ) is a generalization of the test operator in the standard (regular) propositional dynamic logic [20]. So $?\varphi := ?(\varphi,\varphi)$ enables us to check if both elements of a given pair satisfies φ. Moreover, the program construction $\pi \upharpoonright \mathsf{j}$ allows us to execute a program π by keeping agent j for the input and the output world-agent pairs. Together with other program constructions, the program construction $\pi \upharpoonright \mathsf{j}$ is useful for providing the axiom system for the preference changing operations to be introduced in Section 4. The underlying idea of this use is the following: the compounded program $(\leq\upharpoonright \mathsf{j})\cap(\leq\upharpoonright \mathsf{k})$ enables us to formalize the *distributed* preferences between agents j and k.

The following two definitions establish what a model is and how formulas of $\mathcal{HL}$ are interpreted over them.

Definition 3 (*PR*-model)**.** A *PR* model is a tuple $M = (F,V)$ where $F = (W,\{\leq_i,\preccurlyeq_i\}_{i\in A})$ is a *PR*-frame and V is a valuation function from $\mathsf{P}\cup\mathsf{N}_1\cup\mathsf{N}_2$ to $\mathcal{P}(W\times A)$ assigning a subset of the form $\{w\}\times A$ to a world-nominal $\mathsf{a}\in\mathsf{N}_1$ and a subset of the form $W\times\{i\}$ to an agent-nominal $\mathsf{i}\in\mathsf{N}_2$. Throughout the paper, we denote the unique element in the first coordinate of $V(\mathsf{a}) = \{w\}\times A$ by $\underline{\mathsf{a}}$ and the second coordinate of $V(\mathsf{i}) = W\times\{i\}$ by $\underline{\mathsf{i}}$ (see Figure 3).

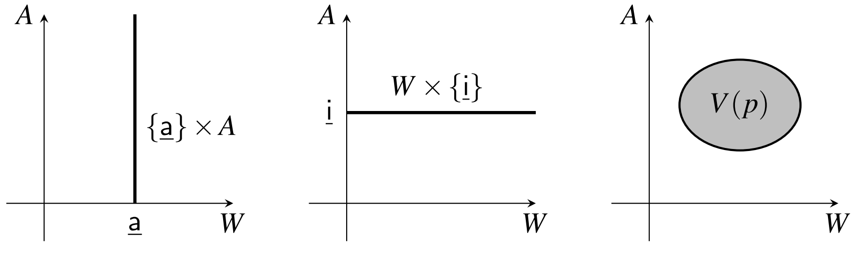

Figure 3: Valuation in *PR* model

Definition 4 (Truth definition)**.** Given a *PR*-model M, the satisfaction relation $M,(w,i)\Vdash\varphi$ (read: “agent i satisfies φ at w in M”) and relations $R_\pi\subseteq(W\times A)^2$ are defined by simultaneous induction as in Table 1. We say that φ is *valid* in a *PR*-model M (written: $M\Vdash\varphi$) if $M,(w,i)\Vdash\varphi$ for all pairs (w,i) in M and that φ is *valid* in a *PR*-frame F if φ is valid in the *PR*-models (F,V) for all valuations V.

Table 1: Satisfaction Relation $M,(w,i) \Vdash \varphi$ and Interpretation R_π for Program π

Satisfaction Relation:

$$\begin{array}{rcl@{\qquad}rcl}
M,(w,i) \Vdash p & \text{iff} & (w,i) \in V(p), & M,(w,i) \Vdash \mathsf{a} & \text{iff} & w = \underline{\mathsf{a}}, \\
M,(w,i) \Vdash \mathsf{i} & \text{iff} & i = \underline{\mathsf{i}}, & M,(w,i) \Vdash \neg\varphi & \text{iff} & M,(w,i) \not\Vdash \varphi, \\
M,(w,i) \Vdash \varphi \vee \psi & \text{iff} & M,(w,i) \Vdash \varphi \text{ or } M,(w,i) \Vdash \psi, & M,(w,i) \Vdash @_{\mathsf{a}}\varphi & \text{iff} & M,(\underline{\mathsf{a}},i) \Vdash \varphi, \\
M,(w,i) \Vdash @_{\mathsf{i}}\varphi & \text{iff} & M,(w,\underline{\mathsf{i}}) \Vdash \varphi, & M,(w,i) \Vdash \langle\pi\rangle\psi & \text{iff} & (w,i)R_\pi(v,j) \text{ and } M,(v,j) \Vdash \psi, \\
 & & & & & \text{for some } (v,j) \in W \times A.
\end{array}$$

Interpretation of Program:

$$\begin{array}{rcl@{\qquad}rcl}
(w,i)R_{\leq}(v,j) & \text{iff} & w \leq_i v \text{ and } i = j, & (w,i)R_{1_W}(v,j) & \text{iff} & w,v \in W \text{ and } i = j, \\
(w,i)R_{\sqsubseteq_{\mathsf{k}}}(v,j) & \text{iff} & w = v \text{ and } i \preccurlyeq_{\underline{\mathsf{k}}} j, & (w,i)R_{1_A}(v,j) & \text{iff} & w = v \text{ and } i,j \in A, \\
(w,i)R_{-\pi}(v,j) & \text{iff} & ((w,i),(v,j)) \notin R_\pi, & (w,i)R_{\pi^{-1}}(v,j) & \text{iff} & (v,j)R_\pi(w,i), \\
(w,i)R_{\pi\cup\rho}(v,j) & \text{iff} & (w,i)R_\pi(v,j) \text{ or } (w,i)R_\rho(v,j), & (w,i)R_{\pi\cap\rho}(v,j) & \text{iff} & (w,i)R_\pi(v,j) \text{ and } (w,i)R_\rho(v,j), \\
(w,i)R_{\pi\restriction\mathsf{k}}(v,j) & \text{iff} & (w,\underline{\mathsf{k}})R_\pi(v,\underline{\mathsf{k}}) & (w,i)R_{?(\varphi,\psi)}(v,j) & \text{iff} & M,(w,i) \Vdash \varphi \text{ and } M,(v,j) \Vdash \psi.
\end{array}$$

Definition 5. As for relational expressions, we define $-_W\pi$ and $-_A\pi$ as $-\pi \cap 1_A$ and $-\pi \cap 1_W$, respectively. Moreover, we define a program $\geq$ and $<$ as $\leq^{-1}$ and $\leq \cap -_W \geq$, respectively, and a program $\sqsupseteq_{\mathsf{k}}$ and $\sqsubset_{\mathsf{k}}$ as $\sqsubseteq_{\mathsf{k}}^{-1}$ and $\sqsubseteq_{\mathsf{k}} \cap -_A \sqsupseteq_{\mathsf{k}}$, respectively. Futhermore, we define a program $\simeq$ as $\leq \cap \geq$ and a program $=_{\mathsf{k}}$ as $\sqsubseteq_{\mathsf{k}} \cap \sqsupseteq_{\mathsf{k}}$, respectively. Finally, $?\varphi$ is defined as the program term $?(\varphi,\varphi)$.

The logic $\mathcal{HL}$ also have the global modalities $[1_W]$ and $[1_A]$ whose truth conditions can be written as follows:

$$M,(w,i) \Vdash [1_W]\psi \quad \text{iff} \quad \text{For all } v \in W, M,(v,i) \Vdash \psi,$$

$$M,(w,i) \Vdash [1_A]\psi \quad \text{iff} \quad \text{For all } j \in A, M,(w,j) \Vdash \psi.$$

It is remarked that

$$(w,i)R_{-_W\pi}(v,j) \quad \text{iff} \quad w = v \text{ and } ((w,i),(w,j)) \notin R_\pi; \quad (w,i)R_{-_A\pi}(v,j) \quad \text{iff} \quad i = j \text{ and } ((w,i),(v,i)) \notin R_\pi,$$

since $-_W\pi := -\pi \cap 1_A$ and $-_A\pi := -\pi \cap 1_W$. When $\alpha \in \{\leq, \geq\}$ and $\beta \in \{\sqsubseteq_{\mathsf{k}}, \sqsupseteq_{\mathsf{k}} \mid \mathsf{k} \in \mathsf{N}_2\}$, we can rewrite the semantic clauses as follows:

$$\begin{array}{lcl}
M,(w,i) \Vdash \langle\alpha\rangle\psi & \text{iff} & \text{For some } v \in W, (w,i)R_\alpha(v,i) \text{ and } M,(v,i) \Vdash \psi. \\
M,(w,i) \Vdash \langle -_W\alpha\rangle\psi & \text{iff} & \text{For some } v \in W, ((w,i),(v,i)) \notin R_\alpha \text{ and } M,(v,i) \Vdash \psi. \\
M,(w,i) \Vdash \langle\beta\rangle\psi & \text{iff} & \text{For some } j \in A, (w,i)R_\beta(w,j) \text{ and } M,(w,j) \Vdash \psi. \\
M,(w,i) \Vdash \langle -_A\beta\rangle\psi & \text{iff} & \text{For some } j \in A, ((w,i),(w,j)) \notin R_\beta \text{ and } M,(w,j) \Vdash \psi.
\end{array}$$

Following the idea found in [5], we can define the *conditional reliability operator* $R_{\mathsf{k}}(\psi,\varphi)$ (read "the most reliable ψ-agents for agent k satisfy φ.") by

$$R_{\mathsf{k}}(\psi,\varphi) := [1_A](\psi \to \langle\sqsubseteq_{\mathsf{k}}\rangle(\psi \wedge [\sqsubseteq_{\mathsf{k}}](\psi \to \varphi))).$$

The reader can understand this definition as follows (cf. [5, p.103]): if $R_{\mathsf{k}}(\psi,\varphi)$ holds at (w,i), then for every agent j satisfying ψ at w, there must exist an even more reliable agent (from k's perspective) satisfying ψ at w and the conditional property $\psi \to \varphi$ is satisfied in w for all agents who are more reliable (from k's perspective). The unconditional version $R_{\mathsf{k}}\varphi$ of $R_{\mathsf{k}}(\psi,\varphi)$ is defined as $R_{\mathsf{k}}(\top,\varphi)$ which can be read as "the most reliable agents for agent k satisfy φ." We may also define the "diamond"-version of $R_{\mathsf{k}}\varphi$ as $\neg R_{\mathsf{k}}\neg\varphi$, denoted by $\langle R_{\mathsf{k}}\rangle\varphi$. Then $\langle R_{\mathsf{k}}\rangle\mathsf{j}$ means that agent j is one of the most reliable agents for k.

We can also formalize the notion of agents' *conditional goals* as follows. We define the *conditional goal operator* $G(\psi,\varphi)$ ("the current agent has a goal that she satisfies φ under the condition that she satisfies ψ")[2] by

$$G(\psi,\varphi) := [1_W](\psi \to \langle\leq\rangle(\psi \wedge [\leq](\psi \to \varphi))).$$

Similarly to $R_{\mathsf{k}}(\psi,\varphi)$ above, the reader can understand this definition as follows (cf. [5, p.103]): if $G(\psi,\varphi)$ holds at (w,i), then for every world v where agent i satisfies ψ, there must exist an even more preferred world u where i satisfies ψ and i also satisfies the conditional property $\psi \to \varphi$ at all the more preferred worlds than u. Then the unconditional goal operator $G\varphi$ is defined as $G(\top,\varphi)$, which read as "the current agent's goal that she satisfies is φ."

Example 2. Let us represent "the current agent puts Monet's picture on wall x" by a world-nominal a_x in the setting of Example 1. On the *PR*-frames F_1^{ex} and F_2^{ex} of Example 1, we define $V(\mathsf{a}_x) = \{(w_x,i),(w_x,j),(w_x,k)\}$ where $x = l$ or r. We use i, j, k as syntactic names (i.e., agent nominals) for i, j and k, where we interpret, e.g., $\underline{\mathsf{i}} = i$ in terms of our valuation function V. Define $M_n^{ex} := (F_n^{ex}, V)$ where n is 1 or 2. For example, the preference $w_l <_i w_r$ can be formalized as a formula $@_{\mathsf{i}}@_{\mathsf{a}_l}\langle<\rangle\mathsf{a}_r$, which is valid on M_n^{ex} for both $n = 1$ and 2. We can formalize Isabella's reliability of $i \prec_i j$ as $@_{\mathsf{i}}\langle\sqsubset_{\mathsf{i}}\rangle\mathsf{j}$ which is valid on M_n^{ex} for both $n = 1$ and 2. When $n = 1$, the following formula is valid on M_1^{ex} ($k \prec_i i \prec_i j$; $k \prec_j j \prec_j i$; $i \prec_k k$, $j \prec_k k$):

$$@_{\mathsf{k}}\langle\sqsubset_{\mathsf{i}}\rangle\mathsf{i} \wedge @_{\mathsf{i}}\langle\sqsubset_{\mathsf{i}}\rangle\mathsf{j} \wedge @_{\mathsf{k}}\langle\sqsubset_{\mathsf{j}}\rangle\mathsf{j} \wedge @_{\mathsf{j}}\langle\sqsubset_{\mathsf{j}}\rangle\mathsf{i} \wedge @_{\mathsf{i}}\langle\sqsubset_{\mathsf{k}}\rangle\mathsf{k} \wedge @_{\mathsf{j}}\langle\sqsubset_{\mathsf{k}}\rangle\mathsf{k}.$$

When $n = 2$, instead of the conjuncts $@_{\mathsf{k}}\langle\sqsubset_{\mathsf{i}}\rangle\mathsf{i}$ and $@_{\mathsf{k}}\langle\sqsubset_{\mathsf{j}}\rangle\mathsf{j}$ of the above formula, we have the following valid formula on M_2^{ex} ($i \prec_i j$; $j \prec_j i$; $i \prec_k k$, $j \prec_k k$):

$$@_{\mathsf{k}}\langle -_A(\sqsubseteq_{\mathsf{i}} \cup \sqsupseteq_{\mathsf{i}})\rangle\mathsf{i} \wedge @_{\mathsf{k}}\langle -_A(\sqsubseteq_{\mathsf{i}} \cup \sqsupseteq_{\mathsf{i}})\rangle\mathsf{j} \wedge @_{\mathsf{k}}\langle -_A(\sqsubseteq_{\mathsf{j}} \cup \sqsupseteq_{\mathsf{j}})\rangle\mathsf{i} \wedge @_{\mathsf{k}}\langle -_A(\sqsubseteq_{\mathsf{j}} \cup \sqsupseteq_{\mathsf{j}})\rangle\mathsf{j},$$

which says that Ken is not connected to Isabella and John for both Isabella's and John's reliability. Moreover, $@_{\mathsf{i}}G\mathsf{a}_x$ formalizes "Isabella has a goal that she puts Monet's picture on wall x" and, when $x = r$, $@_{\mathsf{i}}G\mathsf{a}_r$ is valid on M_n^{ex} for all n. Similarly, $@_{\mathsf{j}}G\mathsf{a}_l$ and $@_{\mathsf{k}}G\mathsf{a}_r$ are also valid in M_n^{ex} for all n. For M_1^{ex}, we can see that, from Isabella's perspective, John is the most reliable agent whose goal is given by a_l. This can be formalized as $R_{\mathsf{i}}(G\mathsf{a}_l,\mathsf{j})$, which is valid on M_1^{ex}. For M_2^{ex}, we can see that, from Isabella's perspective, Ken and John are both maximally reliable agents. Moreover, John and Ken have the goals of putting the picture on the left and right wall, respectively. These facts can be formalized as $R_{\mathsf{i}}(G\mathsf{a}_l,\mathsf{j})$ and $R_{\mathsf{i}}(G\mathsf{a}_r,\mathsf{k})$, which are valid on M_2^{ex}.

2.2 Soundness and completeness

The static axiom systems H**PR** and H**PR**$_{(m,n)}$ (where m and n are fixed natural numbers more than 0) for two-dimensional hybrid logics of preference and reliability are given as in Table 2, where *uniform substitution* means a substitution that uniformly replaces propositional variables by formulas and nominals from N_i by nominals from N_i ($i = 1$ or 2). For example, $@_{\mathsf{b}}@_{\mathsf{k}}(\varphi \to [\leq](\mathsf{a} \wedge \mathsf{i}))$ is a uniform substitution instance of $@_{\mathsf{a}}@_{\mathsf{i}}(p \to [\leq]q)$.

Let us comments on axioms and inference rules of the static axiom systems H**PR** and H**PR**$_{(m,n)}$. Bi-hybrid logical axioms and inference rules of H**PR** are axioms and inference rules for modal operators

[2]The authors owe this point to Thomas Ågotnes.

Table 2: Axiomatizations H**PR** and H$\mathbf{PR}_{(m,n)}$

Bi-Hybrid Logical Axioms of H**PR**
All classical tautologies $\quad$ $(\mathtt{Dual}_\pi)$ $\langle\pi\rangle p \leftrightarrow \neg[\pi]\neg p$ $\quad$ $(\mathtt{K}_\pi)$ $[\pi](p\to q)\to([\pi]p\to[\pi]q)$
Let $\mathsf{n}\in\mathsf{N}_i$ and $(\mathsf{n},\mathsf{m})\in\mathsf{N}_i^2$ $(i=1,2)$ below in this group
$(\mathtt{K}_{@})$ $@_\mathsf{n}(p\to q)\to(@_\mathsf{n}p\to @_\mathsf{n}q)$ $\quad$ $(\mathtt{SelfDual}_{@})$ $\neg @_\mathsf{n}p\leftrightarrow @_\mathsf{n}\neg p$ $\quad$ $(\mathtt{Ref})$ $@_\mathsf{n}\mathsf{n}$
$(\mathtt{Intro})$ $\mathsf{n}\wedge p\to @_\mathsf{n}p$ $\quad$ $(\mathtt{Agree})$ $@_\mathsf{n}@_\mathsf{m}p\to @_\mathsf{m}p$ $\quad$ $(\mathtt{Back})$ $\langle\pi\rangle @_\mathsf{a}@_\mathsf{i}p\to @_\mathsf{a}@_\mathsf{i}p$
Inference Rules of HPR****
Modus Ponens, Uniform Substitutions, Necessitation Rules for $[\pi]$, $@_\mathsf{i}$ and $@_\mathsf{a}$
$(\mathtt{Name})$ From $@_\mathsf{n}\varphi$ infer φ, where $\mathsf{n}\in\mathsf{N}_1\cup\mathsf{N}_2$ is fresh in φ
$(\mathtt{BG}_\pi)$ From $@_\mathsf{a}@_\mathsf{i}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j})\to @_\mathsf{b}@_\mathsf{j}\varphi$ infer $@_\mathsf{a}@_\mathsf{i}[\pi]\varphi$, where b and j are fresh in $@_\mathsf{a}@_\mathsf{i}[\pi]\varphi$
Interaction Axioms of HPR****
$(\mathtt{Com}@)$ $@_\mathsf{i}@_\mathsf{a}p\leftrightarrow @_\mathsf{a}@_\mathsf{i}p$ $\quad$ $(\mathtt{Red}@_1)$ $\mathsf{a}\leftrightarrow @_\mathsf{i}\mathsf{a}$ $\quad$ $(\mathtt{Red}@_2)$ $\mathsf{i}\leftrightarrow @_\mathsf{a}\mathsf{i}$
$(\mathtt{Dcom}\langle W\rangle @_2)$ $@_\mathsf{i}\langle\leq\rangle p\leftrightarrow @_\mathsf{i}\langle\leq\rangle @_\mathsf{i}p$ $\quad$ $(\mathtt{Com}\langle W\rangle @_2)$ $@_\mathsf{i}\langle 1_W\rangle p\leftrightarrow\langle 1_W\rangle @_\mathsf{i}p$
$(\mathtt{Com}\langle A\rangle @_1)$ $@_\mathsf{a}\langle\beta\rangle p\leftrightarrow\langle\beta\rangle @_\mathsf{a}p$ $\quad$ $(\beta\in\{\sqsubseteq_\mathsf{k},1_A\})$
Axioms for Atomic Programs of HPR****
(1_W) $@_\mathsf{a}\langle 1_W\rangle\mathsf{b}$ $\quad$ (1_A) $@_\mathsf{i}\langle 1_A\rangle\mathsf{j}$ $\quad$ $(\mathtt{Eq}_\sqsubseteq)$ $@_\mathsf{i}\mathsf{j}\to([\sqsubseteq_\mathsf{i}]p\leftrightarrow[\sqsubseteq_\mathsf{j}]p)$
$(4_\leq)$ $@_\mathsf{a}\langle\leq\rangle\mathsf{b}\wedge @_\mathsf{b}\langle\leq\rangle\mathsf{c}\to @_\mathsf{a}\langle\leq\rangle\mathsf{c}$ $\quad$ $(\mathtt{Ref}_\leq)$ $@_\mathsf{a}\langle\leq\rangle\mathsf{a}$
$(4_\sqsubseteq)$ $@_\mathsf{j}\langle\sqsubseteq_\mathsf{i}\rangle\mathsf{k}\wedge @_\mathsf{k}\langle\sqsubseteq_\mathsf{i}\rangle\mathsf{l}\to @_\mathsf{j}\langle\sqsubseteq_\mathsf{i}\rangle\mathsf{l}$ $\quad$ $(\mathtt{Ref}_\sqsubseteq)$ $@_\mathsf{j}\langle\sqsubseteq_\mathsf{i}\rangle\mathsf{j}$
Axioms for Compounded Programs of HPR****
$(-)$ $@_\mathsf{a}@_\mathsf{i}\langle-\pi\rangle(\mathsf{b}\wedge\mathsf{j})\leftrightarrow @_\mathsf{a}@_\mathsf{i}\neg\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j})$ $\quad$ $(\mathtt{Conv})$ $@_\mathsf{a}@_\mathsf{i}\langle\pi^{-1}\rangle(\mathsf{b}\wedge\mathsf{j})\leftrightarrow @_\mathsf{b}@_\mathsf{j}\langle\pi\rangle(\mathsf{a}\wedge\mathsf{i})$
$(\cup)$ $\langle\pi\cup\rho\rangle p\leftrightarrow\langle\pi\rangle p\vee\langle\rho\rangle p$ $\quad$ $(\cap)$ $@_\mathsf{a}@_\mathsf{i}\langle\pi\cap\rho\rangle(\mathsf{b}\wedge\mathsf{j})\leftrightarrow @_\mathsf{a}@_\mathsf{i}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j})\wedge @_\mathsf{a}@_\mathsf{i}\langle\rho\rangle(\mathsf{b}\wedge\mathsf{j})$
$(\pi\restriction\mathsf{k})$ $@_\mathsf{a}@_\mathsf{i}\langle\pi\restriction\mathsf{k}\rangle(\mathsf{b}\wedge\mathsf{j})\leftrightarrow @_\mathsf{a}@_\mathsf{k}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{k})$ $\quad$ $(?)$ $@_\mathsf{a}@_\mathsf{i}\langle ?(\varphi,\psi)\rangle(\mathsf{b}\wedge\mathsf{j})\leftrightarrow @_\mathsf{a}@_\mathsf{i}\varphi\wedge @_\mathsf{b}@_\mathsf{j}\psi$
Additional Axioms and Rules for H$\mathbf{PR}_{(m,n)}$
$(\lvert W\rvert\leq m)$ $\bigvee_{0\leq k\neq l\leq m}@_{\mathsf{a}_k}\mathsf{a}_l$ $\quad$ $(\lvert A\rvert\leq n)$ $\bigvee_{0\leq k\neq l\leq n}@_{\mathsf{i}_k}\mathsf{i}_l$
$(\lvert W\rvert\geq m)$ From $\left(\bigwedge_{1\leq k\neq l\leq m}\neg @_{\mathsf{a}_k}\mathsf{a}_l\right)\to\psi$ infer ψ, where a_ks are fresh in ψ
$(\lvert A\rvert\geq n)$ From $\left(\bigwedge_{1\leq k\neq l\leq n}\neg @_{\mathsf{i}_k}\mathsf{i}_l\right)\to\psi$ infer ψ, where i_ks are fresh in ψ.

$[\pi]$ indexed by a program π and satisfaction operators $@_\mathsf{i}$ and $@_\mathsf{a}$. They are almost the same as the axiomatization of one-dimensional hybrid logic (see, e.g., [4, 1]), though we employ the axiom $(\mathtt{Back})$ as the two-dimensional version of the ordinary one dimensional axiom, say, $\langle\leq\rangle @_\mathsf{a}p\to @_\mathsf{a}p$.

Interaction Axioms of H**PR** are axioms to capture the interaction between agent-dimension and world-dimension. For example, the axioms $(\mathtt{Red}@_1)$ and $(\mathtt{Red}@_2)$ reflect the idea of nominals as a "vertical line" $\{\underline{\mathsf{a}}\}\times A$ and a "horizontal line" $W\times\{\underline{\mathsf{i}}\}$ over two-dimensional "plane" $W\times A$, respectively. $(\mathtt{Dcom}\langle W\rangle @_2)$, $(\mathtt{Com}\langle W\rangle @_2)$ and $(\mathtt{Com}\langle A\rangle @_1)$ states how atomic programs interact with satisfaction operators.

Axioms for atomic programs of H**PR** states our required conditions for *PR*-frames, i.e., $\leq_i$ and $\preccurlyeq_i$ are both preorders. The axiom $(\mathtt{Eq}_\sqsubseteq)$ is needed to take care of the case when distinct agent-nominals designate just one agent. Axioms for compounded programs H**PR** enables us to capture the definition of them inside the axiomatization.

Finally, the additional axioms and rules for H$\mathbf{PR}_{(m,n)}$ enables us to capture the restriction to a fixed finite cardinality on a set W of worlds and a set A of agents.

Let us move to soundness of H**PR** and H$\mathbf{PR}_{(m,n)}$ for our intended semantics based on *PR*-models.

Theorem 1 (Soundness). 1. If a formula φ is derivable in $\mathsf{H}\mathbf{PR}$, then φ is valid in all (possibly infinite agents and/or worlds) *PR*-models.

2. If a formula φ is derivable in $\mathsf{H}\mathbf{PR}_{(m,n)}$ then φ is valid in all *PR*-models with fixed m worlds and fixed n agents.

Proof. For soundness, we only prove the validity of some non-trivial axioms. Fix any *PR*-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i\in A}, V)$ and any pair (w,i). For the axiom $(\pi \upharpoonright \mathsf{k})$, we proceed as follows:

$$
\begin{aligned}
M,(w,i) \Vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle \pi \upharpoonright \mathsf{k}\rangle(\mathsf{b}\wedge \mathsf{j}) \quad &\text{iff} \quad M,(\underline{\mathsf{a}},\underline{\mathsf{i}}) \Vdash \langle \pi \upharpoonright \mathsf{k}\rangle(\mathsf{b}\wedge\mathsf{j}),\\
&\text{iff} \quad (\underline{\mathsf{a}},\underline{\mathsf{i}})R_{\pi\upharpoonright\mathsf{k}}(\underline{\mathsf{b}},\underline{\mathsf{j}}),\\
&\text{iff} \quad (\underline{\mathsf{a}},\underline{\mathsf{k}})R_{\pi}(\underline{\mathsf{b}},\underline{\mathsf{k}}),\\
&\text{iff} \quad M,(w,i) \Vdash @_{\mathsf{a}}@_{\mathsf{k}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{k}).
\end{aligned}
$$

Next, we move to the axiom $(\mathtt{Dcom}\langle W\rangle @_2)$. We proceed as follows:

$$
\begin{aligned}
M,(w,i) \Vdash @_{\mathsf{i}}\langle\leq\rangle p \quad &\text{iff} \quad M,(w,\underline{\mathsf{i}}) \Vdash \langle\leq\rangle p,\\
&\text{iff} \quad \text{For some } v\in W,\ wR_{\leq}v \text{ and } M,(v,\underline{\mathsf{i}}) \Vdash p,\\
&\text{iff} \quad \text{For some } v\in W,\ wR_{\leq}v \text{ and } M,(v,\underline{\mathsf{i}}) \Vdash @_{\mathsf{i}}p,\\
&\text{iff} \quad M,(w,i) \Vdash @_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{i}}p.
\end{aligned}
$$

Finally, we move to $(\mathtt{Com}\langle A\rangle @_1)$. Let $\beta \in \{\sqsubseteq_{\mathsf{k}}, 1_A\}$. The validity is checked as follows:

$$
\begin{aligned}
M,(w,i) \Vdash @_{\mathsf{a}}\langle\beta\rangle p \quad &\text{iff} \quad M,(\underline{\mathsf{a}},i) \Vdash \langle\beta\rangle p,\\
&\text{iff} \quad \text{For some } j\in A,\ (w,i)R_{\beta}(w,j) \text{ and } M,(\underline{\mathsf{a}},j) \Vdash p,\\
&\text{iff} \quad \text{For some } j\in A\ (w,i)R_{\beta}(w,j) \text{ and } M,(w,j) \Vdash @_{\mathsf{a}}p,\\
&\text{iff} \quad M,(w,i) \Vdash \langle\beta\rangle @_{\mathsf{a}}p.
\end{aligned}
$$

□

For semantic completeness of $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$ for our intended semantics based on *PR*-frames, we prove the strong completeness results for the restricted class of *PR*-models, i.e., *named PR*-models (cf.[1] or [3, p.437]).

Definition 6 (Named Model). Let $F = (W, \{\leq_i, \preccurlyeq_i\}_{i\in A})$ be *PR*-frame and V be a valuation function. We say that $M = (F,V)$ is *named* if every pair $(w,i) \in W\times A$ there exists a pair $(\mathsf{a},\mathsf{i}) \in \mathsf{N}_1 \times \mathsf{N}_2$ of nominals such that $w = \underline{\mathsf{a}}$ and $i = \underline{\mathsf{i}}$.

Definition 7. We say that a set Δ of formulas define a class $\mathbb{F}$ of possibly infinite *PR*-frames if, for every *PR*-frame F, the following equivalence holds: $F \in \mathbb{F}$ iff all formulas $\psi \in \Gamma$ are valid in F. A set Δ of formulas is *satisfiable* in a class $\mathbb{F}$ of possibly infinite *PR*-frames if there exists a *PR*-frame F, a valuation V on F and a pair (w,i) in F such that $(F,V),(w,i) \models \psi$ for all $\psi \in \Gamma$.

Definition 8 (Pure Formula). We say that a formula φ is *pure* if it does not contain any propositional variables.

Theorem 2 (Strong Completeness). Let Δ be a set of pure formulas and $\mathsf{H}\mathbf{PR}\Delta$ and $\mathsf{H}\mathbf{PR}_{(m,n)}\Delta$ be axiomatic expansions of $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$ by Δ, respectively.

1. If a formula set Σ is consistent in $\mathsf{H}\mathbf{PR}\Delta$ then Σ is satisfiable in the class $\mathbb{F}$ defined by Σ of (possibly infinite) *PR*-frames.

2. If a formula set Σ is consistent in $\mathsf{H}\mathbf{PR}_{(m,n)}\Delta$ then Σ is satisfiable in the class $\mathbb{F}$ defined by Σ of *PR*-frames with fixed m worlds and fixed n agents.

Proof. (Outline) We provide the outline of arguments for both items. The reader who is interested in the detail can check Appendix A for proofs. The axiomatization of bi-hybrid logic (the first two groups of Table 2) with interaction axioms of Table 2 extended with additional axioms having no propositional variables (known as *pure axioms*) is shown to be strongly complete for semantic structures called *d*-product of Kripke frames by the strategy from [28, Theorem 4.12]. We may extend the argument to cover named *PR* models by taking care of: (1) axioms in the third and fourth groups that characterize atomic and compounded programs: (2) axioms in the fifth group that characterize that $\leq_i$ and $\preccurlyeq_i$ are pre-orders. The undelying *PR* frame of the named *PR* model is also shown to be an element of the intended frame class defined by Δ, since Δ is a set of pure formulas. As for the completeness of $\mathsf{H}\mathbf{PR}_{(m,n)}$, we can combine the above argument with Henkin construction in [4] (which allows us to ensure the namedness of a countermodel) for the new inference rules ($|W| \geq m$) and ($|A| \geq n$). ☐

Corollary 1 (Soundness and Completeness)**.** A formula φ is valid in all (possibly infinite agents and/or worlds) *PR*-models iff φ is derivable in $\mathsf{H}\mathbf{PR}$. Moreover, φ is valid in all *PR*-models with fixed m worlds and fixed n agents iff φ is derivable in $\mathsf{H}\mathbf{PR}_{(m,n)}$.

For $\mathsf{H}\mathbf{PR}_{(m,n)}$, we can obtain the following from Corollary 1.

Corollary 2. $\mathsf{H}\mathbf{PR}_{(m,n)}$ is decidable.

We note that, as far as the authors know, decidability is still unknown for $\mathsf{H}\mathbf{PR}$, even for the fragment of $\mathsf{H}\mathbf{PR}$ without program constructions (cf. [28]). So, related computational properties of such fragment has not been yet well-studied (for purely bi-modal logic fragment with a slightly different semantics, the reader is referred to [26]).

3 Reliability change operations

Now that we have proposed our static logical language to discuss about preference and reliability changes let us move on to the dynamics. Some relevant questions in the realm of preference and reliability are the following: Under what conditions can preferences and reliabilities of a particular agent change? What are the effects of those changes? In this section, we will concentrate on the first question from the reliability point of view, which is the main focus of this paper. [3] The preference perspective will be dealt with in the next section. With the proposed two-dimensional hybrid language we will also describe the reliability change operators. We are modelling all theses issues from a deliberation viewpoint and as such we assume that all the agents' publicly announce their preferences (cf. our running example in Section 1) - the study of reliability dynamics in this section and the study of preference dynamics in the next section both deal with *effects* of such announcements, not the announcements themselves. Thus the study will provide answers to the second question as well.

[3]Our motivating example describes the process from reliability change on an announcement of preferences to preference change based on the updated reliability. One reason why we put the reliability change first is to emphasize the novelty of this paper, i.e., to demonstrate how a new reliability relation is formed based on agents' preferences. This does not mean, however, that preference is more fundamental than reliability. However, it is important to distinguish between the absolute and comparative judgements of reliability, and it is remarked that our notion of reliability is assumed to be comparative. Our position in this paper is rather neutral, i.e., we treat reliability and preference on a par. But, it would be interesting to investigate and characterize possible relationships between reliability and preference in terms of our two dynamics.

There is an inherent connection between preference and reliability. On some occasions, we rely more on those whose preferences align with us. In some other cases, it may well happen that we do not rely on our opinion (preferences) too much and hence find somebody more reliable who has a different opinion. For example, if it is about betting on a game, one would rely on preferences of an expert player much more than one's own intuition, even if these preferences are quite the opposite.

3.1 Defining Reliability Dynamics

In what follows, the notion of "matching preference orders" will form the basis for modelling reliability dynamics. The idea is that two preference orderings match each other to a certain extent if the orderings are identical on some subset of the state space W. A full match indicates that the orderings coincide on the whole domain; a partial match indicates that they coincide up to some proper subset of the domain.

Definition 9 (Matching preferences)**.** Let $F = (W, \{\leq_i, \preccurlyeq_i\}_{i\in A})$ be a *PR* frame and let $i, j \in A$ be an agent. If $\leq_i$ is identical with $\leq_j$ on $W' \subseteq W$, then W' is said to be a *set of matches* (or a *match set*) for i and j (notation: $\leq_i \sim_{W'} \leq_j$). We denote the set of all match sets between agents i and j by $M_{ij} \subseteq \wp(W)$.

- Preference orders $\leq_i$ and $\leq_j$ are said to *fully match* each other iff $\leq_i \sim_W \leq_j$, that is $W \in M_{ij}$.[4] We use $FullMat(i)$ to denote the set of agents in A having full match with i.

- Preference orders $\leq_i$ and $\leq_j$ have *zero match* with each other iff there is no $W' \subseteq W$ with $|W'| \geq 2$ such that $\leq_i \sim_{W'} \leq_j$, that is $W' \in M_{ij}$.[5] We denote by $ZeroMat(i)$ the set of agents in A having zero match with i.

- Take any $i \in A$. For each $j(\neq i)$, let M^{max}_{ij} (or M^{min}_{ij}) denote the set of sets of match whose cardinality is maximum (or minimum) among all the match sets between i and j. We note that all the elements of M^{max}_{ij} (or M^{min}_{ij}) has the same cardinality. We may use $\|M^{max}_{ij}\|$ (or $\|M^{min}_{ij}\|$) to mean the maximum (or minimum) cardinality. Let $x_i = max\{\|M^{max}_{ij}\| : j \in A\setminus\{i\}\}$ ($y_i = min\{\|M^{min}_{ij}\| : j \in A\setminus\{i\}\}$).

  By $MaxMat(i)$ (or $MinMat(i)$) we denote the set of all agents j in $A\setminus\{i\}$ such that $\|M^{max}_{ij}\| = x_i$ ($\|M^{min}_{ij}\| = y_i$). This generalises the notion of full (or zero) match, as in the full match (or, zero match) case, $M^{max}_{ij} = \{W\}$ (M^{min}_{ij} is the set of all singleton subsets of W, i.e., $\{\{w\} \mid w \in W\}$, respectively).[6]

  We note that all the elements of M^{max}_{ij} (or M^{min}_{ij}) has the same cardinality. We may use $\|M^{max}_{ij}\|$ (or $\|M^{min}_{ij}\|$) to mean the maximum (or minimum) cardinality. Let $x = max\{\|M^{max}_{ij}\| : j \in A\setminus\{i\}\}$ ($y = min\{\|M^{min}_{ij}\| : j \in A\setminus\{i\}\}$).

We have that the definitions provided above is very general in nature so as to capture the fact that $\|M^{max}_{ij}\| = \|M^{max}_{ik}\|$ (or, $\|M^{min}_{ij}\| = \|M^{min}_{ik}\|$) does not imply $j = k$. In addition, with minor modifications, we can also capture the set of agents having matches equal to/greater than/less than a relevant finite number n, e.g., $n = \frac{|W|}{2}$.

Let us now give some simple examples of such matches. In our running example, as described in Section 1, Isabella has full match with Ken, whereas John has zero match with Isabella and Ken. For the other kinds of matches, e.g. maximal matches, say, one can consider the example of a 10-member expert panel interviewing a group of 25 candidates in several rounds for a single available teaching position.

[4]Note how, by the finiteness of W (the reflexivity of the preference relations), there is always a maximal $X \subseteq W$ such that $\leq_i \sim_X \leq_j$ ($X \in M_{ij}$) for all agents i, j.

[5]For the same reason, there is always a minimal $X \subseteq W$ such that $\leq_i \sim_X \leq_j$ ($X \in M_{ij}$) for all agents i, j.

[6]Then $\|M^{max}_{ij}\|$ should not be confused with $|M^{max}_{ij}|$, i.e., the cardinality of M^{max}_{ij}. For example, $M^{max}_{ij} = \{W\}$ and $\|M^{max}_{ij}\| = |W|$ when we consider full match and $\{\{w\} \mid w \in W\}$ and so $\|M^{min}_{ij}\| = 1$ when we consider zero match.

After the first round, each of the experts provide a preference list of the candidates and accordingly, expert i sees that the maximal number of matching with her ordering is, say, 5, and each of the experts $\{j,k,l\}$ has a maximal set of match with i. With these 'matching' definitions above, we are ready to define some operations for reliability change.

Definition 10 (Full, Zero, Maximal, Minimal matching upgrade)**.** Agent i puts those agents that have full/zero/maximal/minimal match with her own preference ordering above those that do not, keeping her old reliability ordering within each of the two zones. More precisely, if $\preccurlyeq_i$ is agent i's current reliability ordering, then her new reliability ordering $\preccurlyeq_i'$ is defined by:

$$j \preccurlyeq_i' k \text{ iff } \big(j,k \in V \wedge j \preccurlyeq_i k\big) \vee \big(j \notin V \wedge k \in V\big) \vee \big(j,k \notin V \wedge j \preccurlyeq_i k\big),$$

where $V = \mathit{FullMat}(i)$, $\mathit{ZeroMat}(i)$, $\mathit{MaxMat}(i)$ and $\mathit{MinMat}(i)$, respectively.

The definition above considers all kinds of possibilities while introducing the notion of matching upgrades. On one hand we consider the natural case that an agent who has full match with agent i's own preference ordering is more reliable than an agent who has zero match, given by the notion of full matching upgrade. On the other hand, zero matching upgrade considers exactly the opposite, that is, an agent who has zero match with agent i's own preference ordering is more reliable than an agent who has full match. The latter case might come up when an agent needs to consider preferences of certain experts, where the concerned agent is aware that he or she has no expertise at all.

Note that the reliability orderings are in general pre-orderings. So, even if both j and k belong to some V corresponding to some i, if they are not related by their original ordering, they will not be related by the new ordering. The only difference that these changing proposals are making is that, if for some j and k, one is in the corresponding V and the other is not, then we introduce the relevant ordering between them, even if they were unrelated previously. We should also mention here that even if we consider reliability orderings as total pre-orders, the same definition could be applied to define the corresponding upgrades, as done in [15]. We will use both total and non-total pre-orders for reliability orderings while defining preference upgrade operators in Section 4.

Before providing a general definition of these upgrades, we provide some examples of these operations based on the examples of matches provided above. In our running example (cf. Section 1), the change in Isabella's reliability orderings can be attributed to a 'full matching upgrade', whereas the change in John's reliability ordering can be attributed to a 'zero matching upgrade'. In the example of a 10-member expert panel interviewing a group of 25 candidates in several rounds for a single available teaching position, by maximal match upgrade operation, i can consider j,k,l to be more reliable than the others in terms of their choices at the start of the next round of interviews.

The following general definition encompasses the ones given in Definition 10. The idea is to define, for each agent i, an equivalence relation on A and a total ordering $\sqsubseteq$ between the generated equivalence classes, say $V_1 \sqsubseteq \cdots \sqsubseteq V_t$. Then, a new reliability ordering can be defined by using this $\sqsubseteq$ relation across equivalent classes and by following the old ordering inside each equivalence class.

Definition 11 (Equivalence)**.** Let $\equiv_i$ be an equivalence relation on A generating equivalence classes $V_1,\ldots,V_t$, and let $V_1 \sqsubseteq \cdots \sqsubseteq V_t$. Then we define: $j \preccurlyeq_i' k$ iff

$$\Big(j \preccurlyeq_i k \;\wedge\; \bigvee\nolimits_{\ell=1}^{t} \{j,k\} \subseteq V_\ell\Big) \;\vee\; \bigvee\nolimits_{\ell>1} \Big(j \in \bigcup\nolimits_{m<\ell} V_m \;\wedge\; k \in V_\ell\Big).$$

Proposition 1. The updated reliability ordering $\preccurlyeq_i'$ (for each i) given above preserves reflexivity and transitivity and hence, is a pre-ordering. In case the reliability ordering $\preccurlyeq_i$ is a total pre-order for each i, then the updated reliability ordering $\preccurlyeq_i'$ would also be a total pre-ordering.

Proof. Follows from the definition, as properties of the old reliability orderings get transferred to the new ordering within each equivalence classes, and in between the equivalence classes it is decided by the total ordering $\sqsubseteq$. $\square$

We have seen that, given an equivalence relation on the agent set A, we can define a new reliability ordering for each agent, based on her old reliability ordering and a total ordering among those equivalence classes. Now the next question is: how to construct from a given 'set' of preference matches such equivalence relation which will give our desired possibilities of reliability change. Let us start with our running example. Recall that $W = \{w_l, w_r\}$. Let us consider the set $\mathbb{S}_i := \{\{\{w_l\},\{w_r\}\},\{\{w_l,w_r\}\}\}$ (the index "i" for "Isabella"). Then the element $\{\{w_l\},\{w_r\}\}$ of $\mathbb{S}_i$ gives us the matching set of Isabella with John and the remaining element $\{\{w_l,w_r\}\}$ of $\mathbb{S}_i$ gives us the matching set of Isabella with Ken. The following definition provides a general way to set up at least two equivalence relations from (possibly a subset of) $\mathbb{S}_i$.

Definition 12 (Equivalence based on sets of matches)**.** Let $\mathbb{S}_i \subseteq \wp\wp(W)$ be a non-empty set of sets of subsets of W ($\mathbb{S}_i$ is called the *base set* below) for each $i \in A$, satisfying the conditions,

- all members of a element $\mathcal{S} \in \mathbb{S}_i$ have the same cardinality, i.e., for any $\mathcal{S} \in \mathbb{S}_i$ and any $X, Y \in \mathcal{S}$, $|X| = |Y|$ holds.
- for any two distinct sets $\mathcal{S}, \mathcal{T} \in \mathbb{S}_i$, if $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then A and B have different cardinalities.

Let $|\mathbb{S}_i|$ denote cardinality of $\mathbb{S}_i$. For the sake of convenience let us enumerate the members of $\mathbb{S}_i$, with $\mathbb{S}_i[k]$ its kth element.

For any two agents $e, f \in A$, define $e \equiv_i^{max} f$ (or $e \equiv_i^{min} f$) iff either of the following holds:

1. there exists $X, Y \in \mathbb{S}_i[k]$ for some k such that $X \in M_{ie}^{max}$ (or $X \in M_{ie}^{min}$), and $Y \in M_{if}^{max}$ (or $Y \in M_{if}^{min}$);
2. for all k, $\mathbb{S}_i[k] \cap M_{ie}^{max} = \mathbb{S}_i[k] \cap M_{if}^{max} = \varnothing$ (or $\mathbb{S}_i[k] \cap M_{ie}^{min} = \mathbb{S}_i[k] \cap M_{if}^{min} = \varnothing$, respectively).

Note that these relations also depend on the choice of $\mathbb{S}_i$ – the definition of individual reliability upgrade operators differs in this aspect. The corresponding reliability upgrades are termed uniformly as *match reliability upgrade*.

Let us first explore the idea behind the definition above. We have considered sets of subsets of W as our basis. Given an agent i, each such set of subsets corresponds to possible sets of preference matches with other agents. Thus we need the condition of the 'same cardinality'. Note that if there exists at least one agent other than i whose set of matches with i is the whole set W, then maximal matching upgrade reduces to full matching upgrade for that agent i. Similarly, if there exists at least one agent who has zero match with i, then minimal matching upgrade reduces to zero matching upgrade for that agent i. As an immediate consequence we have the following:

Proposition 2. $\equiv_i^{max}$ and $\equiv_i^{min}$ are equivalence relations.

Proof. The conditions of reflexivity and symmetry are immediate from the definition. Let us now prove the transitivity of $\equiv_i^{max}$. Transitivity of $\equiv_i^{min}$ will follow in the same way. Let $e \equiv_i^{max} f$ and $f \equiv_i^{max} g$. We need to show that $e \equiv_i^{max} g$. Case I: Suppose for all k, $\mathbb{S}_i[k] \cap M_{ie}^{max} = \varnothing$. Then, by condition (2), it follows that $\mathbb{S}_i[k] \cap M_{if}^{max} = \varnothing$ and $\mathbb{S}_i[k] \cap M_{ig}^{max} = \varnothing$. Hence, $e \equiv_i^{max} g$. Case II: Suppose there exists $X \in \mathbb{S}_i[k]$ for some k such that $X \in M_{ie}^{max}$. Since all members of M_{ie}^{max} have the same cardinality, and all members of M_{if}^{max} have the same cardinality, our assumption $e \equiv_i^{max} f$ enables us to find a $Y \in \mathbb{S}_i[k]$ such that $Y \in M_{if}^{max}$ by condition (1). We also note that we can use the same index k here. Since all members of M_{if}^{max} have the same cardinality, and all members of M_{ig}^{max} have the same cardinality, and $f \equiv_i^{max} g$ holds, we also have a $Z \in \mathbb{S}_i[k]$ such that $Z \in M_{ig}^{max}$. Thus $e \equiv_i^{max} g$. This completes the proof. $\square$

Given the equivalence classes formed by the relations defined above, one can propose some intuitive ordering among them to define the relevant reliability change operators (in this sense it is noted that such an intuitive ordering between equivalence classes is not induced but assumed). The operators defined earlier (cf. Definition 10) are particular instances of this *match reliability upgrade* with the following table summarizing each of the operators. We note that the set(s) of matches for agent i, say, described below come from a member ($\mathbb{S}_i[k]$, say) of a set $\mathbb{S}_i$ that we have defined earlier:

	Sets of matches and equivalence relations	Induced equivalence classes with orderings
Full	$(\{W\}, \equiv_i^{max})$	$A \setminus FullMat(i) \sqsubseteq FullMat(i)$
Zero	$(\{\{w\} \mid w \in W\}, \equiv_i^{min})$	$A \setminus ZeroMat(i) \sqsubseteq ZeroMat(i)$
Max	$(M_{ij_1}^{max} \cup \cdots \cup M_{ij_m}^{max}, \equiv_i^{max})$	$A \setminus MaxMat(i) \sqsubseteq MaxMat(i)$
Min	$(M_{ij_1}^{min} \cup \cdots \cup M_{ij_l}^{min}, \equiv_i^{min})$,	$A \setminus MinMat(i) \sqsubseteq MinMat(i)$

Here, the second column of each row consists of the list of sets of subsets of W, based on which the equivalence relations ($\equiv_i^{max}$ or $\equiv_i^{min}$) are defined, together with the defined relation, and the third column represents a total ordering over the induced equivalence classes. The first pair of each tuple consists of the list of sets of subsets of W based on which the equivalence relations ($\equiv_i^{max}$ or $\equiv_i^{min}$) are defined, together with the defined relation, and this is followed by a total ordering over the induced equivalence classes. To see how this works, let us consider the *full matching upgrade*: the list is given by $\{W\}$, and the equivalence relation $\equiv_i^{max}$ induce the equivalence classes $FullMat(i)$ and $A \setminus FullMat(i)$ over A, and then the required ordering is given by $A \setminus FullMat(i) \sqsubseteq FullMat(i)$. For the case of maximal (minimal) matching upgrade, $\{j_1, \ldots, j_m\} = \{j \in A \setminus \{i\} : \|M_{ij}^{max}\| = x_i\}$ ($\{j_1, \ldots, j_l\} = \{j \in A \setminus \{i\} : \|M_{ij}^{min}\| = y_i\}$). Please recall Definition 9 for a detailed explanation of x_i and y_i, respectively.

Our final consideration is about the way agents can be informed about each others' preferences, so that they can change their reliabilities accordingly. As mentioned earlier, a public announcement of the agents' individual preferences would be a way to go about this, and the reliabilities could be changed depending on the individual preferences according to proposals described above.

3.2 Expressing the reliablity dynamics

To describe reliability dynamics from the previous section, we first introduce one new terminology.

Definition 13. We say that a formula φ is *global* for states if $\{ w \in W \mid M, (w,i) \Vdash \varphi \} = W$ or $\varnothing$ for every *PR*-model $M = (w, \{ \leq_i, \preccurlyeq_i \}_{i \in A})$ and $i \in A$.

For example, a formula of the form $@_{\mathsf{a}}\chi$ (recall: a is a world-nominal) is global for states. Moreover, it is easy to see that a formula φ is global for states iff φ is equivalent to a formula of the form $@_{\mathsf{a}}\chi$. Then the following dynamic operators are added to the static syntax of $\mathcal{HL}$. The new syntax $\mathcal{HL}_{\{\mathrm{rc}\}}$ is defined to be an expansion of $\mathcal{HL}$ with all operators $\langle \mathrm{rc}_{\mathcal{E}}^{\mathsf{i}} \rangle$, where i is an agent-nominal and $\mathcal{E} = (\varphi_k)_{1 \leq k \leq n}$, a finite list of global $\mathcal{HL}$-formulas for states. An underlying semantic intuition for $\langle \mathrm{rc}_{\mathcal{E}}^{\mathsf{i}} \rangle$ is: Given a *PR*-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i \in A}, V)$, the list $\mathcal{E} = (\varphi_k)_{1 \leq k \leq n}$ can be regarded as a partition of A (i.e., an equivalence relation on agents) in the sense that $(\{ i \in A \mid M, (w,i) \Vdash \varphi_k \})_{1 \leq k \leq n}$ (recall that φ_k is global for states, so $\{ i \in A \mid M, (w,i) \Vdash \varphi_k \} = \{ i \in A \mid M, (v,i) \Vdash \varphi_k \}$ for any w and v) forms a partition of A, and the reliability ordering $\preccurlyeq_{\underline{\mathsf{i}}}$ of the original *PR* model M is rewritten into the updated reliability ordering $\preccurlyeq_{\underline{\mathsf{i}}}'$ as in Definition 11 above (the order of the list $\mathcal{E}$ can be regarded as a total ordering $\sqsubseteq$ used in Definition 11).

Definition 14 (Operators)**.** Given any list $\mathcal{E} = (\varphi_k)_{1 \leq k \leq n}$ of global formulas for states, a formula $\mathrm{Eq}(\mathcal{E})$ is defined as the conjunction of $[1_A] \bigvee_{1 \leq k \leq n} \varphi_k$ (exhaustiveness for agents) and $\bigwedge_{1 \leq j \neq k \leq n} \neg \langle 1_A \rangle (\varphi_j \wedge \varphi_k)$ (pairwise disjointness for agents). Given a list $\mathcal{E} = (\varphi_k)_{1 \leq k \leq n}$ and a *PR*-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i \in A}, V)$, define:

$$M, (w,j) \Vdash \langle \mathrm{rc}_{\mathcal{E}}^{\mathsf{i}} \rangle \varphi \quad \text{iff} \quad M, (w,j) \Vdash \mathrm{Eq}(\mathcal{E}) \text{ and } \mathrm{rc}_{\mathcal{E}}^{\mathsf{i}}(M), (w,j) \Vdash \varphi,$$

where $\mathrm{rc}^{\mathsf{i}}_{\mathcal{E}}(M)$ is the same model as M except $\preccurlyeq_{\underline{\mathsf{i}}}$ is replaced by $\preccurlyeq'_{\underline{\mathsf{i}}}$ of Definition 11.

Example 3. We assume that our *PR*-models under consideration are always named (recall Definition 6), i.e., for every point (a,i) in a model, there is a pair of a state-nominal and an agent-nominal (a,i) such that $\underline{\mathsf{a}} = a$ and $\underline{\mathsf{i}} = i$. Recall also that a domain W in a *PR*-model is always finite. We assume that we have fixed the number of elements in the domain, say, the number is n. In what follows, we fix a set W of states and a set A of agents and a valuation V on $W \times A$ (this means that preference and reliability relations can be arbitrary but we keep the same state-names for W). Let W be the set of all finite nominals that are names of W and we assume that W and W have the same cardinality.[7] Here let us describe how we can get relational transformer for full and zero match upgrades.

First, we consider the full-match upgrade on W. Fix any agent nominal i and consider the following formula to represent the set $FullMat(\underline{\mathsf{i}})$:

$$\mathsf{FullMat}(\mathsf{i}) := \bigwedge_{\mathsf{a},\mathsf{b}\in\mathsf{W}} \left(@_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b} \leftrightarrow @_{\mathsf{a}}\langle\leq\rangle\mathsf{b}\right).$$

Let us explain how the formula $\mathsf{FullMat}(\mathsf{i})$ encodes the information of $FullMat(\underline{\mathsf{i}})$. The satisfaction of the conjunct of $\mathsf{FullMat}(\mathsf{i})$ at (w,j) is equivalent with: $v \leq_{\mathsf{i}} u$ iff $v \leq_j u$, for every $v,u \in W$. Then our list $\mathcal{E}$ for the full match upgrade is $\mathcal{E} = (\mathsf{FullMat}(\mathsf{i}), \neg\mathsf{FullMat}(\mathsf{i}))$, where we note that all formulas in the list are global for states.

Let us now move to the zero-match upgrade. Fix any agent nominal i and consider the following global formula for states to represent the set $ZeroMat(\underline{\mathsf{i}})$:

$$\mathsf{ZeroMat}(\mathsf{i}) := \neg\mathsf{i} \wedge \neg \bigvee_{\mathsf{W}'\subseteq\mathsf{W},\,|\mathsf{W}'|\geq 2} \left(\bigwedge_{\mathsf{a},\mathsf{b}\in\mathsf{W}'} \left(@_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b} \leftrightarrow @_{\mathsf{a}}\langle\leq\rangle\mathsf{b}\right)\right).$$

By a similar argument as above, we can understand that $\mathsf{ZeroMat}(\mathsf{i})$ does represent the information of $ZeroMat(\underline{\mathsf{i}})$. Then our list $\mathcal{E}$ for the zero match upgrade is $\mathcal{E} = (\mathsf{ZeroMat}(\mathsf{i}), \neg\mathsf{ZeroMat}(\mathsf{i}))$.

It is remarked that we need to have the syntactic information of the underlying set W of states and set A of agents and names of states, but we do not need to describe the information of preference ordering and reliability ordering when we define $\mathsf{FullMat}(\mathsf{i})$ and $\mathsf{ZeroMat}(\mathsf{i})$.

For an axiom system for the modality $\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}}\rangle$, we will provide recursion axioms: valid formulas and validity-preserving rules indicating how to translate a formula with the new modality into a provably equivalent one without it. In this case, the modalities can take the form of any relational expression. So we provide a 'matching' relational expression in the original model M by defining relational transformers similar to those in [16, 17], in spirit of the program transformers of [31].

Definition 15 (Relational transformer)**.** Let $\mathcal{E} = (\varphi_k)_{1\leq k\leq n}$ be a list and $|\mathcal{E}|$ the length n of the list. Define $\mathrm{P}^k_{\mathcal{E}} := \bigvee_{l=1}^{k} \varphi_l$, representing the union of the partitions upto k. A relational transformer $\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}$ is a function from relational expressions to relational expressions defined as follows.

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\alpha) := \alpha \quad (\alpha \in \{\leq, 1_W, 1_A\}),} \\
\multicolumn{3}{l}{\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\sqsubseteq_{\mathsf{i}}) := \left(\sqsubseteq_{\mathsf{i}} \cap \bigcup_{k=1}^{|\mathcal{E}|} ?(\varphi_k,\varphi_k)\right) \cup \left(\bigcup_{k=2}^{|\mathcal{E}|} ?(\mathrm{P}^{k-1}_{\mathcal{E}},\varphi_k)\right),} \\
\multicolumn{3}{l}{\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\sqsubseteq_{\mathsf{k}}) := (?(@_{\mathsf{i}}\mathsf{k}, @_{\mathsf{i}}\mathsf{k}) \cap \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\sqsubseteq_{\mathsf{i}})) \cup (?(\neg @_{\mathsf{i}}\mathsf{k}, \neg @_{\mathsf{i}}\mathsf{k}) \cap \sqsubseteq_{\mathsf{k}}) \quad (\mathsf{k} \neq \mathsf{i}),} \\
\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(-\pi) & := & -\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi), \qquad \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi^{-1}) \; := \; \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi)^{-1}, \\
\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi\cup\rho) & := & \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi) \cup \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\rho), \quad \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi\cap\rho) \; := \; \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi) \cap \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\rho), \\
\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi \upharpoonright \mathsf{j}) & := & \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi) \upharpoonright \mathsf{j}, \qquad \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(?(\varphi,\psi)) \; := \; ?(\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}}\rangle\varphi, \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}}\rangle\psi).
\end{array}$$

[7] Under the assumption here, any *PR*-models whose domain is W validates $@_{\mathsf{a}}\neg\mathsf{b}$ for any distinct nominals $\mathsf{a},\mathsf{b} \in \mathsf{W}$ since all elements of W is named by state-nominals.

An underlying idea for $\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\sqsubseteq_{\mathsf{i}})$ is to mimic Definition 11 in terms of our program constructions. When $\mathsf{k} \neq \mathsf{i}$, i.e., k and i are syntactically distinct agent nominals, the reader may wonder why we have generalized test operators "$?(@_{\mathsf{i}}\mathsf{k}, @_{\mathsf{i}}\mathsf{k})$" and "$?(\neg @_{\mathsf{i}}\mathsf{k}, \neg @_{\mathsf{i}}\mathsf{k})$" in the definition of $\mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\sqsubseteq_{\mathsf{k}})$. This is because the same agent might have two distinct (syntactic) names.

Theorem 3. The axioms and rules below together with those of H**PR** (or, those of $\mathsf{H}\mathbf{PR}_{(m,n)}$) provide sound and complete axiom systems for $\mathcal{HL}_{\{\mathrm{rc}\}}$ with respect to possibly infinite named *PR* models (or, named *PR* models with *m* worlds and *n* agents, respectively).

$$
\begin{array}{llllll}
\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle x & \leftrightarrow & \mathrm{Eq}(\mathcal{E}) \wedge x & \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \neg\varphi & \leftrightarrow & \mathrm{Eq}(\mathcal{E}) \wedge \neg \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \varphi \\
\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle (\varphi \vee \psi) & \leftrightarrow & \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \varphi \vee \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \psi & \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle @_{\mathsf{n}}\varphi & \leftrightarrow & \mathrm{Eq}(\mathcal{E}) \wedge @_{\mathsf{n}} \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \varphi, \\
\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \langle \pi \rangle \varphi & \leftrightarrow & \mathrm{Eq}(\mathcal{E}) \wedge \langle \mathrm{Tr}^{\mathsf{i}}_{\mathcal{E}}(\pi) \rangle \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \varphi, & \multicolumn{3}{l}{\text{From } \varphi \to \psi \text{, we may infer } \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \varphi \to \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle \psi,}
\end{array}
$$

where $x \in \mathsf{P} \cup \mathsf{N}_1 \cup \mathsf{N}_2$ and $\mathsf{n} \in \mathsf{N}_1 \cup \mathsf{N}_2$.

Proof. Soundness of the new axioms are straightforward. Completeness follows from the completeness of the static system H**PR** or $\mathsf{H}\mathbf{PR}_{(\mathbf{m},\mathbf{n})}$(of Corollary 1) (cf. Chapter 7 of [32], for an extensive explanation of this technique). □

Example 4. Recall that the following formula is valid in M_1^{ex} in our running example:

$$\langle R_{\mathsf{i}} \rangle \mathsf{j} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{i} \wedge \langle R_{\mathsf{k}} \rangle \mathsf{k},$$

i.e., John is one of the maximally reliable persons for Isabella, Isabella is one of the maximally reliable persons for John, Ken is one of the maximally reliable persons for himself. As for M_2^{ex} in our running example, the following formula is valid:

$$\langle R_{\mathsf{i}} \rangle \mathsf{j} \wedge \langle R_{\mathsf{i}} \rangle \mathsf{k} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{i} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{k} \wedge \langle R_{\mathsf{k}} \rangle \mathsf{k}.$$

After both Isabella and John know others' preferences, to model the situation in our running example, we assume that Isabella uses full-match reliability change $\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}_i} \rangle$ and that John uses zero-match reliability change $\langle \mathrm{rc}^{\mathsf{j}}_{\mathcal{E}_j} \rangle$ for both models M_1^{ex} and M_2^{ex}. Then for M_1^{ex}, Isabella's reliability is changed into $j \prec'_i k \prec'_i i$ and John's reliability is changed into $j \prec'_j k \prec'_j i$. Thus we have,

$$M_1^{ex} \models \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}_i} \rangle \langle \mathrm{rc}^{\mathsf{j}}_{\mathcal{E}_j} \rangle (\langle R_{\mathsf{i}} \rangle \mathsf{i} \wedge \neg \langle R_{\mathsf{i}} \rangle \mathsf{j} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{i} \wedge \langle R_{\mathsf{k}} \rangle \mathsf{k}),$$

As for M_2^{ex}, Isabella's reliability is changed into $j \prec'_i i$ and $j \prec'_i k$, and John's reliability is changed into $j \prec'_j i$ and $j \prec'_j k$. So we have,

$$M_2^{ex} \models \langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}_i} \rangle \langle \mathrm{rc}^{\mathsf{j}}_{\mathcal{E}_j} \rangle (\langle R_{\mathsf{i}} \rangle \mathsf{i} \wedge \langle R_{\mathsf{i}} \rangle \mathsf{k} \wedge \neg \langle R_{\mathsf{i}} \rangle \mathsf{j} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{i} \wedge \langle R_{\mathsf{j}} \rangle \mathsf{k} \wedge \langle R_{\mathsf{k}} \rangle \mathsf{k}).$$

4 Preference dynamics

To complete our tale of interdependence of preference and reliability, this section deals with changes in the individual preferences based on agent reliabilities induced by public announcements of the agents' preferences. Once again, we should mention here that we do not focus on the formal representation of such announcement, but rather on the formal representation of its effects. For example, an agent might adopt the preferences of the set of agents on whom she relies the most, or might use the strict preferences of her most reliable agents for 'breaking ties' among her equally-preferred zones. In [16] the authors introduced the *general lexicographic upgrade* operation, which creates a preference ordering following a priority list of orderings. We generalize those operations in the following, where we consider the reliability orderings to be (possibly, total) *pre-orders*, rather than being total orders (that is, also anti-symmetric) as they are in [16]. The latter was quite an artificial assumption on agents' reliabilities as an agent might have the same amount of reliability on two different agents. Agent i's preference ordering *after* an announcement, $\leq_i'$, can be defined in terms of the just announced preferences (the agents' preferences *before* the announcement, $\leq_1, \ldots, \leq_n$) and how much i relied on each agent (i's reliability *after* the announcement, $\preccurlyeq_i'$). As discussed in the running example, we consider change of reliabilities and then consider change of preferences based on the new reliability orderings and the old preference orderings.

4.1 Defining Preference Dynamics when Reliability Orderings are Total Pre-orders

We first present some preference upgrade operators inspired by [30, 16], assuming the *totality* or *connectedness* of the reliability orderings, i.e., all $\preccurlyeq_i$s are preorders satisfying the following property: $j \preccurlyeq_i k$ or $k \preccurlyeq_i j$ for all $j, k \in A$.

Definition 16. Let $X \subseteq A$. We define $u \leq_X v$ if $u \leq_k v$ holds for all agents $k \in X$. Moreover, $u <_X v$ if $u \leq_X v$ and $v \not\leq_X u$, and $u \simeq_X v$ if $u \leq_X v$ and $v \leq_X u$.

We are now ready to define our upgrade operations. Recall that $mr(i)$ denotes the set of all maximally reliable agents for i.

Definition 17 (Drastic Upgrade)**.** Agent i takes the preference ordering of her most reliable agents where they agree, and leaves the rest undecided. More precisely, the upgraded ordering $\leq_i'$ is defined by: $u \leq_i' v$ iff $u \leq_{mr(i)} v$.

We should mention here that in our running example we are using drastic upgrade as our preference upgrade operation, and Isabella, John and Ken's preferences are upgraded accordingly after considering their reliability changes. Some other intuitive upgrades are given in the following. Before moving on to them let us note that even though we have implicitly assumed here that the most reliable agents have unanimous preferences, we need not do that. The definitions can also be given with respect to the common part of their respective orderings as we have done in Definition 17. We have made this assumption just for the ease of exposition.

Definition 18 (Radical (Tiebreaker) Upgrade)**.** Agent i takes the preference ordering of her most reliable agents (of her own), and in the zones equally-preferable by her most reliable agents (by her) she uses her old ordering (the strict preference ordering of her most reliable agents). As before, the rest are made undecided. More precisely, the upgraded ordering $\leq_i'$ is defined by:

$$u \leq_i' v \quad \text{iff} \quad u <_{mr(i)} v \vee (u \simeq_{mr(i)} v \wedge u \leq_i v).$$
$$(u \leq_i' v \quad \text{iff} \quad u <_i v \vee (u \simeq_i v \wedge u \leq_{mr(i)} v), \text{ respectively}).$$

For example, when we ask a friend to give us some restaurant suggestions in a new area that we are visiting, we will tend to make our choice among the restaurants that were suggested by the friend. Sometimes, when we are undecided regarding which restaurant to go to among a given list of restaurants we ask a food connoisseur friend to give us his/her choices.

The following definitions of the new preference ordering are more elaborated: they use more than just the current preference orderings of i and those of her most reliable agents. In fact, we use the totality of the reliability orders in formulating these upgrade operators. We need totality to describe the ordering between the different sets of equally-reliable agents, based on which these new preference orderings are defined.

Definition 19 (Lexicographic (Lexicopraohic Tiebreaker) Upgrade)**.** Agent i takes the strict preference ordering of her most reliable agents (her own); within the zones of equally-preferable worlds she uses the strict ordering of her second most reliable agents (the strict preference ordering of her most reliable agents); within the zones of equally-preferable worlds she uses the ordering of her third most reliable agents (the ordering of her second most reliable agents), and so on. More precisely, let $R^i_1, R^i_2, \ldots, R^i_n$ be the mutually disjoint sets of reliable agents given by the pre-order for agent i, with R^i_1 being the set of the most reliable agents for i, and the reliability decreases downwards from 1 to n, the upgraded ordering $\leq'_i$ is defined as:

$$LU : u \leq'_i v \quad \text{iff} \quad u <_{R^i_1} v \vee \left(\bigvee_{1<k<n} \left(\bigwedge_{1 \leq l \leq k-1} (u \simeq_{R^i_l} v) \wedge u <_{R^i_k} v \right) \right) \vee \left(\bigwedge_{1 \leq l \leq n-1} (u \simeq_{R^i_l} v) \wedge u \leq_{R^i_n} v \right).$$

$$LTU : u \leq'_i v \quad \text{iff} \quad u <_i v \vee \left(u \simeq_i v \wedge \left(\bigvee_{1 \leq k<n} \left(\bigwedge_{1 \leq l \leq k-1} (u \simeq_{R^i_l} v) \wedge u <_{R^i_k} v \right) \vee \left(\bigwedge_{1 \leq l \leq n-1} (u \simeq_{R^i_l} v) \wedge u \leq_{R^i_n} v \right) \right) \right).$$

A general lexicographic upgrade operation The upgrades defined so far can be seen as particular instances of a general case in which an ordered list indicates the priority of the preference orderings that are involved in the upgrade.

Definition 20 (General Lexicographic upgrade)**.** A *lexicographic list* $\mathcal{R}$ over W is a finite non-empty list whose elements are sets of indices of preference orderings over W, with $|\mathcal{R}|$ the list's length and $\mathcal{R}[k]$ its kth element ($1 \leq k \leq |\mathcal{R}|$). Put $n := |\mathcal{R}|$ below. Intuitively, $\mathcal{R}$ is a priority list of preference orderings, with $\leq_{\mathcal{R}[1]}$ the one with the highest priority. Given $\mathcal{R}$, the preference ordering $\leq_{\mathcal{R}} \subseteq W \times W$ is defined as

$$u \leq_{\mathcal{R}} v \quad \text{iff} \quad \bigvee_{k=1}^{n-1} \left(u <_{\mathcal{R}[k]} v \ \wedge \ \bigwedge_{l=1}^{k-1} u \simeq_{\mathcal{R}[l]} v \right) \ \vee \ \left(u \leq_{\mathcal{R}[n]} v \ \wedge \ \bigwedge_{k=1}^{n-1} u \simeq_{\mathcal{R}[k]} v \right)$$

Thus, $u \leq_{\mathcal{R}} v$ holds if there is an ordering $\leq_{\mathcal{R}[k]}$ with a strict preference for v over u and all orderings with higher priority see u and v as equally preferred (the first disjunct) if this agrees with the least prioritised ordering ($\leq_{\mathcal{R}[n]}$) and for the rest of them u and v are equally preferred (the second disjunct).

The upgrades defined before are all special cases of this general lexicographic upgrade. In fact, the reader might wonder why to use the list $\mathcal{R}$ when the reliability ordering gives us already an ordering among preference relations. The key is that $\mathcal{R}$ includes only the preference relations (strictly speaking, sets of *indices* of the preference relations) that are actually used when building up the new preference ordering. In this sense, the lexicographic upgrade (Definition 19) is the most natural upgrade since it uses the reliability relation at its fullest (its list $\mathcal{R}$ is exactly i's reliability ordering), but the general lexicographic upgrade allows to work also with other natural upgrades, as the radical or the tie-breaker one. Following [16] we have the following proposition.

Proposition 3. Let $\mathcal{R}$ be a lexicographic list over W. If *every* ordering in $\mathcal{R}[k]$ ($1 \leq k \leq n := |\mathcal{R}|$) is reflexive and transitive, then so is $\leq_{\mathcal{R}}$.

Proof. Reflexivity. Take any $u \in W$. Every $\leq_{\mathcal{R}[k]}$ is reflexive ($1 \leq k \leq n$), so $u \simeq_{\mathcal{R}[k]} u$ for all such k. Hence, by Part 1 of $\leq_{\mathcal{R}}$'s definition, $u \leq_{\mathcal{R}} u$.

Transitivity. Suppose (I) $u \leq_{\mathcal{R}} v$ and (II) $v \leq_{\mathcal{R}} w$. From $\leq_{\mathcal{R}}$'s definition, (I) $u \leq_{\mathcal{R}[n]} v$ and, for all k with $1 \leq k < n$, $u \simeq_{\mathcal{R}[k]} v$, or there is an ordering $k_1 < n$ such that $u <_{\mathcal{R}[k_1]} v$ and, for all k with $1 \leq k < k_1$, $u \simeq_{\mathcal{R}[k]} v$, and (II) $v \leq_{\mathcal{R}[n]} w$ and, for all k with $1 \leq k < n$, $v \simeq_{\mathcal{R}[k]} w$, or there is an ordering $k_2 < n$ such that $v <_{\mathcal{R}[k_2]} w$ and, for all k with $1 \leq k < k_2$, $v \simeq_{\mathcal{R}[k]} w$. We have four cases to consider - here are the second and the fourth.

- Suppose $u \leq_{\mathcal{R}[n]} v$ and, for all k with $1 \leq k < n$, $u \simeq_{\mathcal{R}[k]} v$, and there is $k_2 < n$ such that $v <_{\mathcal{R}[k_2]} w$ and, for all k with $1 \leq k < k_2$, $v \simeq_{\mathcal{R}[k]} w$. First, for all k with $1 \leq k < k_2$, $u \simeq_{\mathcal{R}[k]} w$. Second, let us focus on k_2. On the one hand, since $u \leq_{\mathcal{R}[k]} v$ for all k with $1 \leq k \leq n$, $u \leq_{\mathcal{R}[k_2]} v$. On the other hand, $v <_{\mathcal{R}[k_2]} w$ implies $v \leq_{\mathcal{R}[k_2]} w$. Hence, by the transitivity of $\leq_p$ for each $p \in \mathcal{R}[k_2]$,

$u \leq_{\mathcal{R}[k_2]} w$. Now observe how $w \not\leq_{\mathcal{R}[k_2]} u$; otherwise from $w \leq_{\mathcal{R}[k_2]} u$, $u \leq_{\mathcal{R}[k_2]} v$ and transitivity of $\leq_p$ for each $p \in \mathcal{R}[k_2]$ we would have $w \leq_{\mathcal{R}[k_2]} v$, a contradiction. Then $u <_{\mathcal{R}[k_2]} w$. Thus, by putting the two pieces together, the first disjunct of $\leq_{\mathcal{R}}$'s definition yields $u \leq_{\mathcal{R}} w$.

- Suppose there is $k_1 < n$ such that $u <_{\mathcal{R}[k_1]} v$ (so $u \leq_{\mathcal{R}[k_1]} v$) and, for all k with $1 \leq k < k_1$, $u \simeq_{\mathcal{R}[k]} v$, and there is $k_2 < n$ such that $v <_{\mathcal{R}[k_2]} w$ (so $v \leq_{\mathcal{R}[k_2]} w$) and, for all k with $1 \leq k < k_2$, $v \simeq_{\mathcal{R}[k]} w$.

  Case where $k_1 = k_2$ holds: First observe how, for all k with $1 \leq k < k_1$, $u \simeq_{\mathcal{R}[k]} w$. Second, from the transitivity of $\leq_p$ for each $p \in \mathcal{R}[k_1]$, $u <_{\mathcal{R}[k_1]} w$. Hence, by the first disjunct of the definition, $u \leq_{\mathcal{R}} w$.

  Case where $k_2 > k_1$ holds: k_1's priority is higher than k_2's priority. First observe how, for all k with $1 \leq k < k_1$, $u \simeq_{\mathcal{R}[k]} w$. Second, we focus on k_1. On the one hand, from $u \leq_{\mathcal{R}[k_1]} v$ and $v \leq_{\mathcal{R}[k_1]} w$ (by $k_2 > k_1$), using similar arguments as earlier, we get $u \leq_{\mathcal{R}[k_1]} w$. Moreover, $w \not\leq_{\mathcal{R}[k_1]} u$; otherwise from $w \leq_{\mathcal{R}[k_1]} u$, $v \leq_{\mathcal{R}[k_1]} w$ and transitivity we would have $v \leq_{\mathcal{R}[k_1]} u$, a contradiction. Then, $u <_{\mathcal{R}[k_1]} w$. Hence, by the first disjunct of the definition, $u \leq_{\mathcal{R}} w$.

  Case where $k_1 > k_2$ holds: as the previous one. □

As a consequence of this proposition, the general lexicographic upgrade preserves preorders (and thus our class of semantic models) *when every preference ordering in $\mathcal{R}$ satisfies the requirements*.

Before ending this section we show how the preference change operators defined earlier are particular instances of this *general lexicographic upgrade* with the following table summarizing the lists corresponding to each of the upgrade operators:

Upgrade policies	**Lexicographic lists (cf. Definition 20)**
Drastic Upgrade	$\langle mr(i) \rangle$
Radical Upgrade	$\langle mr(i); \{i\} \rangle$
Tiebreaker Upgrade	$\langle \{i\}; mr(i) \rangle$
Lexicographic Upgrade	$\langle R^i_1; R^i_2; \ldots; R^i_n \rangle$
Lexicographic Tiebreaker Upgrade	$\langle \{i\}; R^i_1; R^i_2; \ldots; R^i_n \rangle$ (Note that the ordering remains same except for $\{i\}$)

4.2 Expressing the Preference Dynamics over Reliability as Total Preordering

Following the ideas of [16], we formalize preference dynamics from the previous section with necessary modifications due to the incorporation of hybrid language. We add the following dynamic operators to the static syntax $\mathcal{HL}$. First of all, we regard all the agents involved in a preference upgrade operator as agent nominals (syntactic names of agents) and so let us denote agent i's syntactic name as i of sans serif and the set of all syntactic names in $mr(i)$ as $\mathsf{mr}(\mathsf{i})$. $\mathcal{HL}_{\{\mathrm{pu}\}}$ is defined to be an expansion of $\mathcal{HL}$ with all operators $\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}} \rangle$, where i be an agent-nominal and $\mathcal{R}$ is a list of sets of agent-nominals defined as in Definition 20. For example, $\mathcal{R} = \mathsf{mr}(\mathsf{i})$ (drastic upgrade) or $\mathcal{R} = (\mathsf{mr}(\mathsf{i}); \{\mathsf{i}\})$ (radical upgrade). We use $\mathsf{N}_2(\mathcal{R})$ to mean the set of all agent nominals occuring in $\mathcal{R}$.

Definition 21 (Operators)**.** A formula $\mathrm{Req}(\mathcal{R})$, representing requirements for the list $\mathcal{R}$ is defined as the conjunction $\bigwedge_{\mathsf{j} \neq \mathsf{k} \in \mathsf{N}_2(\mathcal{R})} \neg @_{\mathsf{j}} \mathsf{k}$ (mutual disjointness of agents involved in $\mathsf{N}_2(\mathcal{R})$). Given a PR-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i \in A}, V)$, define:

$$M, (w, j) \Vdash \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}} \rangle \varphi \quad \text{iff} \quad M, (w, j) \Vdash \mathrm{Req}(\mathcal{R}) \text{ and } \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}(M), (w, j) \Vdash \varphi,$$

where $\mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}(M)$ is the same model as M except $\leq_{\mathsf{i}}$ is replaced by $\leq_{\mathcal{R}}$ as in Definition 20.

Before going into the notion of relational transformer, we have two observations. Firstly, when $\pi := ?(\mathsf{j},\mathsf{j})\cap \leq$, we note that $(w,i)R_\pi(v,k)$ is equivalent to the conjunction of $i = k = \underline{\mathsf{j}}$ and $w \leq_{\underline{\mathsf{j}}} v$. Similarly, when we put $\pi' := ?(\neg\mathsf{j},\neg\mathsf{j})\cap \leq$, we remark that $(w,i)R_{\pi'}(v,k)$ is equivalent to the conjunction of $w \leq_i v$ and $i = k$ and $i \neq \underline{\mathsf{j}}$. Secondly, to reflect the relation $<_X$ in Definition 16, we need our program constructions $\pi \upharpoonright \mathsf{j}$ and $\cap$ to take the intersection of (strict) preference relations of the possibly different agents than i. These observations allow us to capture the idea behind Definitions 17 and 18 syntactically in the following definition.

Definition 22 (Relational transformer)**.** Let us introduce the following abbreviations for relational expressions:

$$\leq_{\mathcal{R}[k]} := \bigcap\{\, \leq\upharpoonright \mathsf{j} \mid \mathsf{j} \in \mathcal{R}[k]\,\}, \quad <_{\mathcal{R}[k]} := \leq_{\mathcal{R}[k]} \cap -_W (\leq^{-1}_{\mathcal{R}[k]}),$$

A *relational transformer* $\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}$ is a function from relational expressions to relational expressions defined as follows.

$$\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\alpha) := \alpha \quad (\alpha \in \{\,1_W, 1_A\,\} \cup \{\, \sqsubseteq_{\mathsf{k}}, \sqsubset_{\mathsf{k}} \mid \mathsf{k} \in \mathsf{N}_2\,\})$$

$$\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\leq) := \Big(?\mathsf{i} \cap \Big(\textstyle\bigcup_{1\leq k\leq n-1}\big(<_{\mathcal{R}[k]} \cap \bigcap_{1\leq l\leq k-1} \simeq_{\mathcal{R}[l]}\big) \cup \big(\leq_{\mathcal{R}[n]} \cap \bigcap_{1\leq k\leq n-1} \simeq_{\mathcal{R}[k]}\big)\Big)\Big) \cup (?\neg\mathsf{i} \cap \leq)$$

$$\begin{array}{llllll}
\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(-\pi) & := & -\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi), & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi^{-1}) & := & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi)^{-1}, \\
\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi \cup \rho) & := & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi) \cup \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\rho), & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi \cap \rho) & := & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi) \cap \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\rho), \\
\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi \upharpoonright \mathsf{j}) & := & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi) \upharpoonright \mathsf{j}. & \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(?(\varphi,\psi)) & := & ?(\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi, \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\psi),
\end{array}$$

where recall that $?\varphi$ is defined as $?(\varphi,\varphi)$.

Example 5. When $\mathcal{R}$ is for the drastic upgrade, i.e., $\mathcal{R} = \mathsf{mr}(\mathsf{i})$, the relation transformer for the atomic program $\leq$ becomes as follows:

$$\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\leq) \quad := \quad \big(?\mathsf{i} \cap \leq_{\mathsf{mr}(\mathsf{i})}\big) \cup (?\neg\mathsf{i} \cap \leq),$$

When $\mathcal{R}$ is for the radical upgrade, i.e., $\mathcal{R} = (\mathsf{mr}(\mathsf{i}); \{\,\mathsf{i}\,\})$, the relation transformer for the atomic program $\leq$ becomes as follows:

$$\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\leq) \quad := \quad \big(?\mathsf{i} \cap (<_{\mathsf{mr}(\mathsf{i})} \cup (\simeq_{\mathsf{mr}(\mathsf{i})} \cap \leq))\big) \cup (?\neg\mathsf{i} \cap \leq)$$

Based on a similar strategy for Theorem 3, we can now prove the following theorem. But, we need to take care of connectedness of *PR*-frames. It is easy to see that $(\mathtt{Conn})$ $@_{\mathsf{i}} \sqsubseteq_{\mathsf{k}} \mathsf{j} \vee @_{\mathsf{j}} \sqsubseteq_{\mathsf{k}} \mathsf{i}$ defines the class of connected *PR*-frames. Let $\mathsf{H}\mathbf{PR}\mathtt{Conn}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}\mathtt{Conn}$ be axiomatic expansions of $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$ by the formula $(\mathtt{Conn})$. It follows from Theorem 2 that these axiomatizations are sound and (strongly) complete for the intended connected *PR*-frame classes, since $(\mathtt{Conn})$ is a pure formulas, i.e., it does not contain any propositional variables.

Theorem 4. The axioms and rules below together with those of $\mathsf{H}\mathbf{PR}\mathtt{Conn}$ (or, those of $\mathsf{H}\mathbf{PR}_{(m,n)}\mathtt{Conn}$) provide sound and complete axiom systems for $\mathcal{HL}_{\{\mathrm{pu}\}}$ with respect to possibly infinite connected *PR* models (or, *PR* models with *m* worlds and *n* agents, respectively).

$$\begin{array}{llllll}
\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle x & \leftrightarrow & \mathrm{Req}(\mathcal{R}) \wedge x & \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\neg\varphi & \leftrightarrow & \mathrm{Req}(\mathcal{R}) \wedge \neg\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi \\
\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle(\varphi\vee\psi) & \leftrightarrow & \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi \vee \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\psi, & \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle @_{\mathsf{n}}\varphi & \leftrightarrow & \mathrm{Req}(\mathcal{R}) \wedge @_{\mathsf{n}}\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi \\
\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\langle\pi\rangle\varphi & \leftrightarrow & \mathrm{Req}(\mathcal{R}) \wedge \langle \mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}(\pi)\rangle\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi, & \multicolumn{3}{l}{\text{From } \varphi\to\psi, \text{ we may infer } \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\varphi \to \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}\rangle\psi,}
\end{array}$$

where $x \in \mathsf{P} \cup \mathsf{N}_1 \cup \mathsf{N}_2$ and $\mathsf{n} \in \mathsf{N}_1 \cup \mathsf{N}_2$.

Example 6. Here we focus on our model M_1^{ex} in Example 2 where the reliablity ordering of each agent is a total preordering. In our running example of Section 1, each agent is regarded to employ drastic upgrades to change his or her preference after Isabella and John's reliability upgrade as in Example 4. In what follows in this example, we consider the expansion of $\mathcal{HL}_{\{\mathrm{rc}\}}$ with all preference upgrade operators $\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}} \rangle$ so that we can use both reliability changing operators and preference upgrade operators in one setting (we note that the expansion does not cause any techinical trouble, i.e., we can easily establish soundness and completeness of the expansion as in Theorem 4). Let us write the corresponding upgrade operators of $\{i, j, k\}$ by $\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}_i} \rangle$ and $\langle \mathrm{pu}^{\mathsf{j}}_{\mathcal{R}_j} \rangle$, $\langle \mathrm{pu}^{\mathsf{k}}_{\mathcal{R}_k} \rangle$, respectively. Then, three flatmates did reach an agreement after all, i.e.,

$$
\begin{aligned}
&@_{\mathsf{i}} G\mathsf{a}_r \wedge @_{\mathsf{j}} G\mathsf{a}_l \wedge @_{\mathsf{k}} G\mathsf{a}_r \wedge \\
&\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}_i} \rangle \langle \mathrm{rc}^{\mathsf{j}}_{\mathcal{E}_j} \rangle \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}_i} \rangle \langle \mathrm{pu}^{\mathsf{j}}_{\mathcal{R}_j} \rangle \langle \mathrm{pu}^{\mathsf{k}}_{\mathcal{R}_k} \rangle (@_{\mathsf{i}} G\mathsf{a}_r \wedge @_{\mathsf{j}} G\mathsf{a}_r \wedge @_{\mathsf{k}} G\mathsf{a}_r).
\end{aligned}
$$

is valid in M_1^{ex}, because upgraded preferences are given by $w_l <'_i w_r$, $w_l <'_j w_r$ and $w_l <'_k w_r$.

Remark 1. We note here that while the main focus of the work is to model joint deliberation in form of simultaneous preference and reliability upgrades, the model operations and modalities of Sections 3.2 and 4.2 deal with single agent upgrades. This presentation style has been chosen in order to simplify notation and readability, but the provided definitions can be easily extended in order to match our goals. In particular, the model operations of Definitions 14 and 21 can be extended to simultaneous upgrades by asking for a list $\mathcal{R}$ of lexicographic lists (with $\mathcal{R}_i$ the list for agent i), and asking for a list $\mathcal{E}$ of partition lists (with $\mathcal{E}_i$ the list for agent i), respectively. Then the corresponding modalities, $\langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}} \rangle$ and $\langle \mathrm{rc}^{\mathsf{i}}_{\mathcal{E}} \rangle$ can still be axiomatised by the presented system with some simple modifications.

4.3 Defining and Expressing Preference Dynamics when Reliability Orderings are Pre-orders

Assuming that the reliability orderings are only pre-orders, we follow [33] in defining relevant preference upgrade operation.

Definition 23 (Comparative Lexicographic Upgrade [33])**.** Agent i agrees with the relative preference ordering that all the agents agree with or else for every agent who does not agree there is a more reliable cluster of agents who agrees. More precisely, the upgraded ordering $\leq'_i$ is defined as follows: $u \leq'_i v$ iff for all $j \in A$, either $u \leq_j v$ or, there exists a $\preceq_i$-cluster $B \neq \varnothing$ with $u \leq_k v$ for all $k \in B$ and $j \prec_i k$ for all $k \in B$. To be more specific, the upgraded ordering $u \leq'_i v$ can be defined as: for all $j \in A$,

$$
u \leq_j v \vee \bigvee_{B \subseteq A} \left(u <_B v \wedge \bigwedge_{k \in B} j \prec_i k \wedge \bigwedge_{k_1, k_2 \in B} k_1 \approx_i k_2 \wedge \bigwedge_{k \in A \setminus B} \bigvee_{k' \in B} k \not\approx_i k' \right).
$$

Proposition 4. If the original preference orderings are pre-orders, the revised orderings are also pre-orders.

The proof of the above proposition follows from the same provided in [33]. Let us comment on how we can express comparative lexicographic upgrade of Definition 23 based on reliability as pre-order between agents. Let us suppose that all the agents in A (we suppose that the number of agents is a fixed n) are named by distinct nominals $\mathsf{i}_1, \ldots, \mathsf{i}_n$. We may regard $\mathcal{R}$ to consist of all the nominals for A itself. Given a PR-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i \in A}, V)$, we define

$$
M, (w, j) \Vdash \langle \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}} \rangle \varphi \quad \text{iff} \quad M, (w, j) \Vdash \mathrm{Req}(\mathcal{R}) \text{ and } \mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}(M), (w, j) \Vdash \varphi,
$$

where $\mathrm{pu}^{\mathsf{i}}_{\mathcal{R}}(M)$ is the same model as M except $\leq_{\mathsf{i}}$ is replaced by $\leq'_{\mathsf{i}}$ as in the above and it is noted that $\mathrm{Req}(\mathcal{R})$ means the mutual disjointness of all the nominals for A. A relational transformer $\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}$ for this comparative lexicographic upgrade is a function from relational expressions to relational expressions defined similarly to $\mathrm{Tp}^{\mathsf{i}}_{\mathcal{R}}$ except:

$$\mathsf{Tp}^{\mathsf{i}}_{\mathcal{R}}(\leq) \quad := \quad (?\mathsf{i} \cap \bigcap_{\mathsf{j} \in A}(\leq \upharpoonright \mathsf{j} \cup \bigcup_{B \subseteq A}(<_B \cap ?\mathsf{cluster}_{\sqsubseteq_{\mathsf{i}}}(B,\mathsf{j})))) \cup (?\neg\mathsf{i} \cap \leq)$$

where it is noted that $\bigcup_{B \subseteq A}$ is a finite union (since A is finite) and a formula $\mathsf{cluster}_{\sqsubseteq_{\mathsf{i}}}(B,\mathsf{j})$ (whose intuitive meaning is "B is a cluster with respect to $\sqsubseteq_{\mathsf{i}}$ and B is above j") is defined as:

$$\bigwedge_{\mathsf{k}_1,\mathsf{k}_2 \in B} @_{\mathsf{k}_1}\langle =_{\mathsf{i}} \rangle \mathsf{k}_2 \wedge \bigwedge_{\mathsf{k} \in A \setminus B} \bigvee_{\mathsf{j} \in B} @_{\mathsf{j}}\langle -_A =_{\mathsf{i}} \rangle \mathsf{k} \wedge \bigwedge_{\mathsf{k} \in B} @_{\mathsf{j}}\langle \sqsubset_{\mathsf{i}} \rangle \mathsf{k}.$$

where recall that $=_{\mathsf{i}} := \sqsubseteq_{\mathsf{i}} \cap \sqsupseteq_{\mathsf{i}}$. Then we can establish soundness and completeness results as in Theorem 4.

Example 7. Here we use the comparative lexicographic upgrade to formalize our running example. We use M_2^{ex} (recall its frame part in Figure 2), since reliability orderings of Isabella and John are pre-orders but not total pre-orders, i.e., k is not connected to i and j in terms of both i's and j's reliability. The reliability orderings of all agents were written as follows:

$$i \prec_i j;\; j \prec_j i;\; i \prec_k k;\; j \prec_k k.$$

The preferences of all agents were given as follows:

$$w_l <_i w_r,\; w_r <_j w_l \text{ and } w_l <_k w_r.$$

In what follows, we first check that the comparative lexicographic upgrade does not give us an agreement in the initial model. Then, after reliability change of Isabella and John as in Example 4, we see that the comparative lexicographic upgrade enables us to reach an agreement.

First, the result of the comparative lexicographic upgrade for all agents' preferences in M_2^{ex} becomes as follows:

$$w_l \not\leq'_i w_r \text{ and } w_r \not\leq'_i w_l,\; w_r <'_j w_l \text{ and } w_l <'_k w_r.$$

So there is no agreement in this stage. Let us comment on Isabella's upgraded preference. When we consider John (j) in Isabella's reliability ordering, $w_l \not\leq_j w_r$ and there is no cluster beyond John in Isabella's reliablity and so we obtain $w_l \not\leq'_i w_r$. In order to see why $w_r \not\leq'_i w_l$ holds, it suffices to note that $w_r \not\leq_k w_l$ and there is no cluster beyond Ken in Isabella's reliablity (recall that Ken is not connected to Isabella and John).

Second, let us consider the model after reliablity change in Example 4 where Isabella uses full-match reliability change and John uses zero-match reliability change for M_2^{ex}. Recall that Isabella's reliability is changed into $j \prec'_i i$ and $j \prec'_i k$, and that John's reliability is changed into $j \prec'_j i$ and $j \prec'_j k$. Let us apply the comparative lexicographic upgrade for all agents' preferences to this model. Then the result is an agreement as follows:

$$w_l <'_i w_r,\; w_l <'_j w_r \text{ and } w_l <'_k w_r.$$

Let us comment on why we obtain $w_l <'_i w_r$. To see that $w_l \leq'_i w_r$, we have $w_l \not\leq_j w_r$ but in Isabella's reliability ordering there is a cluster $\{k\}$ beyond John, and Ken prefers to put the picture on the right wall. By the same argument as above for the initial model, we still have $w_r \not\leq'_i w_l$.

5 Conclusion

This work continues the line of study in [16, 17] and provides a further interplay between the preferences that the agents have about the world around and the reliability attributions they have with respect to one another. We deal with both reliability change based on preferences, and preference change based on reliability, and propose two-dimensional dynamic hybrid logics to express such changes. Consequently, we provide a more detailed formal analysis of decision making processes where such intertwining of preference and reliability considerations are abound (e.g., our running example). In addition to such daily life scenarios, we can also model such connections in collective decision making in multi-agent systems. On the one hand, we have an alternative to preference aggregation models of multi-agent decision making, by considering deliberation among agents regarding their preferences in the line of [16, 17, 33]. On the other hand, we introduce a natural notion of reliability changes that are involved in such decision making processes.

The main technical results that we have are sound and complete axiomatizations which lead to decidability (provided the numbers of agents and of states are *fixed* finite numbers) as well. In process, we also discuss about agents' goals in such situations, e.g. relating reliability attributions with the notions of goals (cf. the running example in the text). The novel contribution of the work is the study of change in reliability attribution of agents based on their preferences.

While we interpret $\leq$ of a *PR*-frame as a preference relation in this paper, we may also interpret it as a plausibility relation and then the corresponding operator to our goal operator $G(\psi,\varphi)$ becomes the relativized belief operabor $B(\psi,\varphi)$, which can be read as "under the condition that ψ, the current agent believes that φ" (cf. [24]). In addition, we could easily equip a plausibility relation into a *PR*-frame to discuss a relationship between preference and belief, which will be a possible topic of further research.

To conclude, let us provide some more pointers towards future work: (1) What other reasonable preference and reliability upgrade policies can there be and how to model them? (2) How to investigate the role of knowledge and/or belief in such changes, especially if manipulation comes into play? (3) What would be the characterizing conditions for reaching consensus in such deliberative processes? We endeavor to provide answers to such questions in future.

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References

[1] Carlos Areces and Balder ten Cate. Hybrid logics. In P. Blackburn, F. Wolter, and J. van Benthem, editors, *Handbook of Modal Logics*, pages 821–868. Elsevier, 2006.

[2] Kenneth J. Arrow, A. K. Sen, and Kotaro Suzumura, editors. *Handbook of Social Choice and Welfare*. Elsevier, 2002. Two volumes.

[3] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[4] Patrick. Blackburn and Balder ten Cate. Pure extensions, proof rules, and hybrid axiomatics. *Studia Logica*, 84(3):277–322, 2006.

[5] C. Boutilier. Conditional logics of normality: A modal approach. *Artificial Intelligence*, 68(1):87–154, 1994.

[6] Burak Can. Weighted distances between preferences. *Journal of Mathematical Economics*, 51:109–115, 2014.

[7] Helen De Cruz. Being friends with a brexiter? Philosophers On. Daily Nous, 2016. `http://dailynous.com/2016/06/28/philosophers-on-brexit/#DeCruz`.

[8] Robert Demolombe. To trust information sources: A proposal for a modal logic framework. In Cristiano Castelfranchi and Yao-Hua Tan, editors, *Trust and Deception in Virtual Societies*, pages 111–134. Kluwer Academic, Dordrecht, 2001.

[9] Robert Demolombe. Reasoning about trust: A formal logical framework. In Christian Damsgaard Jensen, Stefan Poslad, and Theodosis Dimitrakos, editors, *iTrust*, volume 2995 of *Lecture Notes in Computer Science*, pages 291–303. Springer, 2004.

[10] Carmel Domshlak, Eyke Hüllermeier, Souhila Kaci, and Henri Prade. Preferences in AI: An overview. *Artificial Intelligence*, 175(7-8):1037–1052, 2011.

[11] Ulle Endriss. Logic and social choice theory. In A. Gupta and J. van Benthem, editors, *Logic and Philosophy Today*, volume 2, pages 333–377. College Publications, 2011.

[12] Rino Falcone, K. Suzanne Barber, Jordi Sabater-Mir, and Munindar P. Singh, editors. *Trust in Agent Societies, 11th International Workshop, TRUST 2008, Estoril, Portugal, May 12-13, 2008. Revised Selected and Invited Papers*, volume 5396 of *Lecture Notes in Computer Science*. Springer, 2008.

[13] Rino Falcone and Cristiano Castelfranchi. Social trust: A cognitive approach. In Cristiano Castelfranchi and Yao-Hua Tan, editors, *Trust and Deception in Virtual Societies*, pages 55–90. Kluwer Academic, Dordrecht, 2001.

[14] George Gargov, Solomon Passy, and Tinko Tinchev. Modal environment for boolean speculations, preliminary report. In D.G. Skordev, editor, *Mathematical Logic and Its Applications*, pages 253–263. Plenum Press, 1987.

[15] Sujata Ghosh and Katsuhiko Sano. Valuing others' opinions: Preference, belief and reliability dynamics. In A.P. Rocha J. van den Herik and J. Filipe, editors, *Proceedings of the 9th International Conference on Agents and Artificial Intelligence - Volume 2 (ICAART 2017)*, pages 615–623. Scitepress, 2017.

[16] Sujata Ghosh and Fernando R. Velázquez-Quesada. Agreeing to agree: Reaching unanimity via preference dynamics based on reliable agents. In Rafael Bordini, Edith Elkind, Gerhard Weiss, and Pinar Yolum, editors, *AAMAS 2015*, pages 1491–1499, 2015.

[17] Sujata Ghosh and Fernando R. Velázquez-Quesada. A note on reliability-based preference dynamics. In Wiebe van der Hoek, Wesley H. Holliday, and Wen fan Wang, editors, *LORI 2015*, pages 129 –142, 2015.

[18] Alvin I. Goldman. Experts: Which ones should you trust? *Philosophy and Phenomenological Research*, 63(1):85–110, 2001.

[19] Till Grüne-Yanoff and Sven Ove Hansson, editors. *Preference Change*, volume 42 of *Theory and Decision Library*. Springer, 2009.

[20] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic*. MIT Press, Cambridge, MA, 2000.

[21] Andreas Herzig, Emiliano Lorini, Jomi Fred Hübner, and Laurent Vercouter. A logic of trust and reputation. *Logic Journal of the IGPL*, 18(1):214–244, 2010.

[22] Wesley H. Holliday. Trust and the dynamics of testimony. In Lena Kurzen, Davide Grossi, and Fernando R. Velázquez-Quesada, editors, *Logic and Interactive RAtionality. Seminar's yearbook 2009*, pages 118–142. Institute for Logic, Language and Computation, Universiteit van Amsterdam, Amsterdam, The Netherlands, 2010.

[23] Churn-Jung Liau. Belief, information acquisition, and trust in multi-agent systems – a modal logic formulation. *Artificial Intelligence*, 149(1):31–60, 2003.

[24] Fenrong Liu. *Reasoning about Preference Dynamics*, volume 354 of *Synthese Library Series*. Springer, 2011.

[25] Emiliano Lorini, Guifei Jiang, and Laurent Perrussel. Trust-based belief change. In T. Schaub, editor, *ECAI 2014 – 21st European Conference on Artificial Intelligence, 18-22 August 2014, Prague, Czech Republic – Including Prestigious Applications of Intelligent Systems (PAIS 2014)*, volume 263 of *Frontiers in Artificial Intelligence and Applications*, pages 549–554. IOS Press, 2014.

[26] Maarten Marx and Szabolcs Mikulás. Products, or how to create modal logics of high complexity. *Logic Journal of IGPL*, 9(1):71–82, 2001.

[27] Ben Rodenhäuser. *A Matter of Trust: Dynamic Attitudes in Epistemic Logic*. PhD thesis, Institute for Logic, Language and Computation (ILLC), Universiteit van Amsterdam (UvA), Amsterdam, The Netherlands, 2014. ILLC Dissertation Series DS-2014-04.

[28] Katsuhiko Sano. Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic? *Journal of Applied Logic*, 8(4):459–474, December 2010.

[29] Jeremy Seligman, Fenrong Liu, and Patrick Girard. Knowledge, friendship and social announcement. In Johan van Benthem and Fenrong Liu, editors, *Logic Across the University: Foundations and Applications*, volume 47 of *Studies in Logic*, pages 445–469. College Publications, 2013.

[30] Johan van Benthem. Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, 17(2):129–155, 2007.

[31] Johan van Benthem, Jan van Eijck, and Barteld Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.

[32] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic*. Number 337 in Synthese Library. Springer, 2008.

[33] Fernando R. Velázquez-Quesada. Reliability-based preference dynamics: lexicographic upgrade. *Journal of Logic and Computation*, 27(8):2341–2381, 2017.

A Semantic completeness of two-dimensional hybrid logic

This section establishes semantic completeness results of both $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$ possibly expanded with new axioms that do not contain any propositional variables. A semantic completeness result of one-dimensional hybrid logic can be found in, e.g., [4], where we can use nominals as if they are constant symbols in the first-order logic to do a Henkin-style construction to get an appropriate maximal consistent set, from which we can extract a countermodel. As for the semantic completeness result of two-dimensional hybrid logic without program constructions, the reader can refer to the proof in [28, Theorem 4.12]. For the semantics completeness of $\mathsf{H}\mathbf{PR}_{(m,n)}$, we need to combine the argument in [28, Theorem 4.12] and with the Henkin construction in [4] for the new inference rules ($|W| \geq m$) and ($|A| \geq n$). Let us see the detail below.

Remark that we use Λ to denote an axiomatic expansion of our axiomatizations $\mathsf{H}\mathbf{PR}$ or $\mathsf{H}\mathbf{PR}_{(m,n)}$ in what follows in this section. We also remark that the domains W and A of a *PR*-model in this section have possibly infinite members.

First of all, the following proposition is necessary to prove semantic completeness results for $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$.

Proposition 5. Let Λ be one of $\mathsf{H}\mathbf{PR}$ and $\mathsf{H}\mathbf{PR}_{(m,n)}$. Then the following hold.

1. ($\mathtt{Paste}_\pi$) If $\vdash_\Lambda (@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \wedge @_{\mathsf{b}}@_{\mathsf{j}}\varphi) \to \psi$ then $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi \to \psi$ where b and j are fresh in $@_{\mathsf{a}}@_{\mathsf{i}}[\pi]\varphi \to \psi$.
2. $\vdash_\Lambda @_{\mathsf{i}}@_{\mathsf{a}}\mathsf{b} \leftrightarrow @_{\mathsf{a}}\mathsf{b}$.
3. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\mathsf{j} \leftrightarrow @_{\mathsf{i}}\mathsf{j}$.
4. $\vdash_\Lambda @_{\mathsf{n}}\mathsf{m} \to @_{\mathsf{m}}\mathsf{n}$ for $\mathsf{n},\mathsf{m} \in \mathsf{N}_i$ ($i = 1$ or 2).
5. $\vdash_\Lambda @_{\mathsf{n}}\mathsf{m} \wedge @_{\mathsf{m}}\varphi \to @_{\mathsf{n}}\varphi$ for $\mathsf{n},\mathsf{m} \in \mathsf{N}_i$ ($i = 1$ or 2).
6. $\vdash_\Lambda \langle\alpha\rangle @_{\mathsf{a}}p \to @_{\mathsf{a}}p$ where $\alpha \in \{\leq, 1_W\}$.
7. $\vdash_\Lambda \langle\beta\rangle @_{\mathsf{i}}p \to @_{\mathsf{i}}p$ where $\beta \in \{\sqsubseteq_{\mathsf{k}}, 1_A\}$.
8. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b}\wedge\mathsf{j}) \leftrightarrow @_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b} \wedge @_{\mathsf{i}}\mathsf{j}$.
9. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\langle 1_W\rangle(\mathsf{b}\wedge\mathsf{j}) \leftrightarrow @_{\mathsf{a}}\langle 1_W\rangle\mathsf{b} \wedge @_{\mathsf{i}}\mathsf{j}$.
10. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\langle\beta\rangle(\mathsf{b}\wedge\mathsf{j}) \leftrightarrow @_{\mathsf{a}}\mathsf{b} \wedge @_{\mathsf{i}}\langle\beta\rangle\mathsf{j}$ where $\beta \in \{\sqsubseteq_{\mathsf{k}}, 1_A\}$.
11. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\varphi \to (\mathsf{a}\wedge\mathsf{i} \to \varphi)$.
12. $\vdash_\Lambda @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \wedge @_{\mathsf{b}}@_{\mathsf{j}}\varphi \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi$.
13. $\vdash_\Lambda @_{\mathsf{i}}\mathsf{j} \wedge @_{\mathsf{a}}\mathsf{b} \wedge @_{\mathsf{c}}\mathsf{d} \to (@_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{c} \leftrightarrow @_{\mathsf{j}}@_{\mathsf{b}}\langle\leq\rangle\mathsf{d})$.

Proof. (1) Assume that $\vdash (@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \wedge @_{\mathsf{b}}@_{\mathsf{j}}\varphi) \to \psi$ for fresh nominals b and j. Let c and k be fresh nominals in the assumption. Our goals is to show that $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi \to \psi$. It follows from the assumption, ($\mathtt{Agree}$), ($\mathtt{Com@}$) and normality of $@$, we can obtain $\vdash (@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \wedge @_{\mathsf{b}}@_{\mathsf{j}}\varphi) \to @_{\mathsf{c}}@_{\mathsf{k}}\psi$. By ($\mathtt{Agree}$), ($\mathtt{Selfdual}$), ($\mathtt{Com@}$) and normality of $@$, $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \to @_{\mathsf{b}}@_{\mathsf{j}}(\varphi \to @_{\mathsf{c}}@_{\mathsf{k}}\psi)$. By the rule ($\mathtt{BG}_\pi$), we obtain $\vdash @_{\mathsf{a}}@_{\mathsf{i}}[\pi](\varphi \to @_{\mathsf{c}}@_{\mathsf{k}}\psi)$. Since $[\pi]$ is normal, we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}(\langle\pi\rangle\varphi \to \langle\pi\rangle @_{\mathsf{c}}@_{\mathsf{k}}\psi)$. By ($\mathtt{Back}$), $\vdash @_{\mathsf{a}}@_{\mathsf{i}}(\langle\pi\rangle\varphi \to @_{\mathsf{c}}@_{\mathsf{k}}\psi)$ holds, which is equivalent with $\vdash @_{\mathsf{c}}@_{\mathsf{k}}(@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi \to \psi)$. Finally by applying the rule ($\mathtt{Name}$) twice, we conclude that $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi \to \psi$.

(2) By $(\mathtt{Taut})$, we obtain $\vdash @_{\mathsf{a}}\mathsf{b} \leftrightarrow @_{\mathsf{a}}\mathsf{b}$. It follows from $(\mathtt{Nec}_{@})$ and $(\mathtt{K}_{@})$ that $\vdash @_{\mathsf{i}}@_{\mathsf{a}}\mathsf{b} \leftrightarrow @_{\mathsf{i}}@_{\mathsf{a}}\mathsf{b}$. By $(\mathtt{Com@})$, we get $\vdash @_{\mathsf{i}}@_{\mathsf{a}}\mathsf{b} \leftrightarrow @_{\mathsf{a}}@_{\mathsf{i}}\mathsf{b}$. By $(\mathtt{Red@}_1)$, $(\mathtt{Nec})_{@}$ and $(\mathtt{K}_{@})$, $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\mathsf{b} \leftrightarrow @_{\mathsf{a}}\mathsf{b}$. Therefore we conclude $\vdash @_{\mathsf{i}}@_{\mathsf{a}}\mathsf{b} \leftrightarrow @_{\mathsf{a}}\mathsf{b}$.

(3) Similarly shown to item (2).

(4) By $(\mathtt{Intro})$, $\vdash \mathsf{n} \wedge \mathsf{m} \to @_{\mathsf{n}}\mathsf{m}$. It follows from $(\mathtt{Nec}_{@})$ and $(\mathtt{K}_{@})$, $\vdash @_{\mathsf{m}}(\mathsf{n} \wedge \mathsf{m}) \to @_{\mathsf{m}}@_{\mathsf{n}}\mathsf{m}$ holds. Since $@_{\mathsf{m}}$ is normal, $\vdash (@_{\mathsf{m}}\mathsf{n} \wedge @_{\mathsf{m}}\mathsf{m}) \to @_{\mathsf{m}}@_{\mathsf{n}}\mathsf{m}$ holds. By $(\mathtt{Ref})$, we conclude that $\vdash @_{\mathsf{m}}\mathsf{n} \to @_{\mathsf{m}}@_{\mathsf{n}}\mathsf{m}$ by a propositional inference.

(5) By (4), we obtain $\vdash @_{\mathsf{n}}\mathsf{m} \wedge @_{\mathsf{m}}\varphi \to @_{\mathsf{m}}\mathsf{n} \wedge @_{\mathsf{m}}\varphi$. By normality of $@_{\mathsf{m}}$, we get $\vdash @_{\mathsf{n}}\mathsf{m} \wedge @_{\mathsf{m}}\varphi \to @_{\mathsf{m}}(\mathsf{n} \wedge \varphi)$. By $(\mathtt{Intro})$ and the normality of @, we have $\vdash @_{\mathsf{m}}(\mathsf{n} \wedge \varphi) \to @_{\mathsf{m}}@_{\mathsf{n}}\varphi$. It follows that $\vdash @_{\mathsf{n}}\mathsf{m} \wedge @_{\mathsf{m}}\varphi \to @_{\mathsf{m}}@_{\mathsf{n}}\varphi$. Finally we obtain $\vdash @_{\mathsf{n}}\mathsf{m} \wedge @_{\mathsf{m}}\varphi \to @_{\mathsf{n}}\varphi$ by $(\mathtt{Agree})$.

(6) We focus on the case where α is $\leq$. When α is 1_W, we can establish the equivalence by a similar argument as in (7) below. We proceed as follows. By $(\mathtt{Dcom})$, we obtain $\vdash @_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{a}}p \to @_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{i}}@_{\mathsf{a}}p$. By $(\mathtt{Back})$ and $(\mathtt{Agree})$ and normality of @, $\vdash @_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{i}}@_{\mathsf{a}}p \to @_{\mathsf{i}}@_{\mathsf{a}}p$. So we obtain $\vdash @_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{a}}p \to @_{\mathsf{i}}@_{\mathsf{a}}p$. Since $\vdash @_{\mathsf{i}}(\varphi \to \psi) \leftrightarrow (@_{\mathsf{i}}\varphi \to @_{\mathsf{i}}\psi)$, we get $\vdash @_{\mathsf{i}}(\langle\leq\rangle @_{\mathsf{a}}p \to @_{\mathsf{a}}p)$. Finally by $(\mathtt{Name})$ we conclude: $\vdash \langle\leq\rangle @_{\mathsf{a}}p \to @_{\mathsf{a}}p$.

(7) Let $\beta \in \{\sqsubseteq_{\mathsf{k}}, 1_A\}$. Our argument here is similar but simpler than that for (6). By $(\mathtt{Com}\langle A\rangle @_1)$, we have $\vdash @_{\mathsf{a}}\langle\beta\rangle @_{\mathsf{i}}p \to \langle\beta\rangle @_{\mathsf{a}}@_{\mathsf{i}}p$. By (Back) and a propositional inference, we obtain $\vdash @_{\mathsf{a}}\langle\beta\rangle @_{\mathsf{i}}p \to @_{\mathsf{a}}@_{\mathsf{i}}p$. Since $\vdash @_{\mathsf{a}}(\varphi \to \psi) \leftrightarrow (@_{\mathsf{a}}\varphi \to @_{\mathsf{a}}\psi)$, we get $\vdash @_{\mathsf{a}}(\langle\beta\rangle @_{\mathsf{i}}p \to @_{\mathsf{i}}p)$. By $(\mathtt{Name})$, we conclude $\vdash \langle\beta\rangle @_{\mathsf{i}}p \to @_{\mathsf{i}}p$.

(8) First, we prove the left-to-right implication. Since $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b} \wedge \mathsf{j}) \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{b}$ holds by normality of @s and $\langle\leq\rangle$, it suffices to show that $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b} \wedge \mathsf{j}) \to @_{\mathsf{i}}\mathsf{j}$. We proceed as follows. Similarly as above, we can prove $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b} \wedge \mathsf{j}) \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{j}$. By $(\mathtt{Red@}_2)$ and normality of @ and $\langle\leq\rangle$, we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle @_{\mathsf{b}}\mathsf{j}$. It follows from (6) that $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}@_{\mathsf{b}}\mathsf{j}$. By $(\mathtt{Com@})$, $(\mathtt{Red@}_2)$ and (3), we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{j} \to @_{\mathsf{i}}\mathsf{j}$. This finishies to show the left-to-right implication.

Second, let us establish the right-to-left implication. It suffices to prove $\vdash @_{\mathsf{i}}\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}[\leq]\mathsf{j}$. This is because we can deduce from this that $\vdash @_{\mathsf{i}}\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle\mathsf{b} \wedge @_{\mathsf{a}}@_{\mathsf{i}}[\leq]\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b} \wedge \mathsf{j})$. So let us establish $\vdash @_{\mathsf{i}}\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}[\leq]\mathsf{j}$. By (3) and the dual of (6), we get $\vdash @_{\mathsf{i}}\mathsf{j} \to [\leq]@_{\mathsf{i}}\mathsf{j}$. By normality of @, we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}@_{\mathsf{i}}\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}[\leq]@_{\mathsf{i}}\mathsf{j}$. By $(\mathtt{Dcom}\langle W\rangle @_2)$ and the dual of $(\mathtt{Agree})$, we have that $\vdash @_{\mathsf{i}}\mathsf{j} \to @_{\mathsf{a}}@_{\mathsf{i}}[\leq]\mathsf{j}$, as desired.

(9) Similarly shown to item (8).

(10) Similarly shown to item (8).

(11) By the dual of $(\mathtt{Intro})$, we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}\varphi \to (\mathsf{a} \to @_{\mathsf{i}}\varphi)$. We also get $\vdash @_{\mathsf{i}}\varphi \to (\mathsf{i} \to \varphi)$. By combining these, we get the desired goal.

(12) By (11) and monotonicity of $[\pi]$, we get: $\vdash [\pi]@_{\mathsf{b}}@_{\mathsf{j}}\varphi \to [\pi](\mathsf{b} \wedge \mathsf{j} \to \varphi)$, which implies $\vdash [\pi]@_{\mathsf{b}}@_{\mathsf{j}}\varphi \to (\langle\pi\rangle(\mathsf{b} \wedge \mathsf{j}) \to \langle\pi\rangle\varphi)$ since $[\pi]$ is normal. By the dual $@_{\mathsf{b}}@_{\mathsf{j}}\varphi \to [\pi]@_{\mathsf{b}}@_{\mathsf{j}}\varphi$ of $(\mathtt{Back})$, we have $\vdash @_{\mathsf{b}}@_{\mathsf{j}}\varphi \to (\langle\pi\rangle(\mathsf{b} \wedge \mathsf{j}) \to \langle\pi\rangle\varphi)$. Since "$@_{\mathsf{a}}@_{\mathsf{i}}$" is normal, we get $\vdash @_{\mathsf{a}}@_{\mathsf{i}}@_{\mathsf{b}}@_{\mathsf{j}}\varphi \to (@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b} \wedge \mathsf{j}) \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi)$. By (Agree) and a propositional inference, we conclude that $\vdash @_{\mathsf{b}}@_{\mathsf{j}}\varphi \to (@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b} \wedge \mathsf{j}) \to @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\varphi)$, which is propositionally equivalent to our goal.

(13) By the dual of $(\mathtt{Intro})$, $\vdash @_{\mathsf{c}}\mathsf{d} \to (\mathsf{c} \to \mathsf{d})$. Since $\langle\leq\rangle$ is normal, we get $\vdash [\leq]@_{\mathsf{c}}\mathsf{d} \to (\langle\leq\rangle\mathsf{c} \to \langle\leq\rangle\mathsf{d})$. By $(\mathtt{Back})$, $\vdash @_{\mathsf{c}}\mathsf{d} \to (\langle\leq\rangle\mathsf{c} \to \langle\leq\rangle\mathsf{d})$. By normality of $@_{\mathsf{a}}$ and $(\mathtt{Agree})$, $\vdash @_{\mathsf{c}}\mathsf{d} \to (@_{\mathsf{a}}\langle\leq$

$\rangle c \to @_a\langle\leq\rangle d)$. By item (5), $\vdash @_a b \wedge @_c d \to (@_a\langle\leq\rangle c \to @_b\langle\leq\rangle d)$. By normality of $@_i$, we obtain $\vdash @_i @_a b \wedge @_i @_c d \to (@_i @_a\langle\leq\rangle c \to @_i @_b\langle\leq\rangle d)$. By (2) and (3), $\vdash @_a b \wedge @_c d \to (@_i @_a\langle\leq\rangle c \to @_i @_b\langle\leq\rangle d)$. Finally (5) implies that $\vdash @_i j \wedge @_a b \wedge @_c d \to (@_i @_a\langle\leq\rangle c \to @_j @_b\langle\leq\rangle d)$. $\square$

Definition 24. We say that a set Γ of formula is Λ-*inconsistent* if there is a finite subset Γ' of Γ such that $\vdash_\Lambda \bigwedge\Gamma' \to \bot$ and that Γ is Λ-*consistent* if Γ is *not* Λ-inconsistent. A set Γ of formulas is said to be *maximally* Λ-*consistent* (Λ-*MCS*) if Γ is Λ-consistent and complete in the following sense: $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for any formula φ.

Definition 25. We say that Λ is *strongly complete* for a class $\mathbb{F}$ of *PR*-frames if every Λ-consistent set Δ of formulas is satisfiable in a model whose frame is an element of $\mathbb{F}$, i.e., there exists a *PR*-model $M = (W, \{\leq_i, \preccurlyeq_i\}_{i\in A}, V)$ from $\mathbb{F}$ and a pair (w,i) such that $M,(w,i) \Vdash \varphi$ for all formulas $\varphi \in \Delta$.

Definition 26. We define $\mathbb{M}_{\text{all}}$ to be the class of all *PR*-models and $\mathbb{M}_{(m,n)}$ to be the class of all *PR*-models $M = (W, \{\leq_i, \preccurlyeq_i\}_{i\in A}, V)$ such that $|W| = m$ and $|A| = n$.

Definition 27. Let Δ be a set of formulas.

- Δ is said to be *named* if $a \wedge i \in \Delta$ for some state-nominal a and some agent-nominal i;
- Δ is π-*saturated* if $@_a @_i\langle\pi\rangle\varphi \in \Delta$, then $@_a @_i\langle\pi\rangle(b \wedge j) \in \Delta$ and $@_b @_j \varphi \in \Delta$ for some state-nominal b and agent-nominal j;
- Δ *contains at least m distinct states* if $\bigwedge_{1\leq k\neq l\leq m} \neg @_{a_k} a_l \in \Delta$ for some distinct m state-nominals $a_1, \ldots, a_m$.
- Δ *contains at least n distinct agents* if $\bigwedge_{1\leq k\neq l\leq n} \neg @_{i_k} i_l \in \Delta$ for some distinct n agent-nominals $i_1, \ldots, i_n$.

Lemma 1 (Lindenbaum Lemma)**.**

1. When Λ is an axiomatic expansion of $\mathsf{H}\mathbf{PR}$, every Λ-consistent set Δ of formulas can be extended to a maximally Λ-consistent set Δ^+ such that $\Delta \subseteq \Delta^+$ and Δ^+ is named and π-saturated.
2. When Λ is an axiomatic expansion of $\mathsf{H}\mathbf{PR}_{(m,n)}$, every Λ-consistent set Δ of formulas can be extended to a maximally Λ-consistent set Δ^+ such that $\Delta \subseteq \Delta^+$ and Δ^+ is named, π-saturated and contains both at least m-distinct states and at least n-distinct agents.

Proof. First of all, we note, in our proof below, that the rules $(\mathtt{Name})$ and $(\mathtt{BG}_\pi)$ are needed to ensure the properties (named) and (π-saturated), respectively. We prove item 2 alone, since an argument for item 2 contains a necessary argument for item 1. So let Λ be $\mathsf{H}\mathbf{PR}_{(m,n)}$. Let N_1^+ and N_2^+ be fresh countably infinite state- and agent-nominals, respectively. We also suppose that $(b_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$ are enumerations for N_1^+ and N_2^+, respectively. We denote the expanded syntax with N_1^+ and N_2^+ by $\mathcal{HL}^+$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be an enumeration of all formulas in $\mathcal{HL}^+$. In what follows, we inductively construct an increasing sequence $(\Delta_n)_{n\in\mathbb{N}}$ of Λ-consistent sets of formulas as follows.

(Basis) Since N_1^+ and N_2^+ are fresh nominals, we define Δ_0 as follows:

$$\Delta_0 := \Delta \cup \{\, a \wedge i, \bigwedge\nolimits_{1\leq k\neq l\leq m} \neg @_{a_k} a_l, \bigwedge\nolimits_{1\leq k\neq l\leq n} \neg @_{i_k} i_l \,\}$$

where a, a_ks are taken from N_1^+ and i, i_ks are taken from N_2^+. Then the the Λ-consistency of Δ_0 is assured by the rules $(\mathtt{Name})$, $(|W| \geq m)$ and $(|A| \geq n)$.

(Inductive Step) Suppose that we have constructed a finite increasing sequence $(\Delta_k)_{0\leqslant k\leqslant n}$ of Λ-consistent sets of formulas. Depending on the shape of the formula φ_n in our enumeration, we define Δ_{n+1} as follows.

– If $\Delta_n \cup \{\varphi_n\}$ is Λ-consistent and φ_n is of the form $@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\psi$,

$$\Delta_{n+1} := \Delta_n \cup \{\varphi_n, @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}), @_{\mathsf{b}}@_{\mathsf{j}}\varphi\},$$

where b and j are fresh nominals from N_1^+ and N_2^+, respectively.

– If $\Delta_n \cup \{\varphi_n\}$ is Λ-consistent and φ_n is not of the form $@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\psi$,

$$\Delta_{n+1} := \Delta_n \cup \{\varphi_n\}.$$

– Otherwise, we define:

$$\Delta_{n+1} := \Delta_n \cup \{\neg\varphi_n\}.$$

We can prove that Δ_{n+1} is Λ-consistent by the consistency of Δ_n and the rule given in Proposition 5 (1), which is a derived rule from $(\mathtt{BG}_\pi)$.

Finally, we define $\Delta^+ := \bigcup_{n\in\mathbb{N}}\Delta_n$, which can be easily shown to satisfy all the desired properties. □

Definition 28. Let Λ be an axiomatic expansion of $\mathsf{H}\mathbf{PR}$ or $\mathsf{H}\mathbf{PR}_{(m,n)}$ and Σ be π-saturated Λ-consistent set of formulas. The *Henkin-style* Λ*-canonical model*

$$M^\Sigma_\Lambda = (W^\Sigma, A^\Sigma, (\leq^\Sigma_{[\mathsf{i}]}, \preccurlyeq^\Sigma_{[\mathsf{i}]})_{[\mathsf{i}]\in A^\Sigma}, V^\Sigma)$$

is defined as follows:

- $W^\Sigma := \{\,|\mathsf{a}| \mid \mathsf{a}$ is a state-nominal $\}$ where $|\mathsf{a}| := \{\mathsf{b} \mid @_{\mathsf{a}}\mathsf{b}\in\Sigma\}$.
- $A^\Sigma := \{\,[\mathsf{i}] \mid \mathsf{i}$ is an agent-nominal $\}$ where $[\mathsf{i}] := \{\mathsf{j} \mid @_{\mathsf{i}}\mathsf{j}\in\Sigma\}$.
- $|\mathsf{a}| \leq^\Sigma_{[\mathsf{i}]} |\mathsf{b}|$ iff $@_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b}\in\Sigma$.
- $[\mathsf{j}] \preccurlyeq^\Sigma_{[\mathsf{i}]} [\mathsf{k}]$ iff $@_{\mathsf{j}}\langle\sqsubseteq_{\mathsf{i}}\rangle\mathsf{k}\in\Sigma$.
- $(|\mathsf{a}|, [\mathsf{i}]) \in V^\Sigma(\varphi)$ iff $@_{\mathsf{a}}@_{\mathsf{i}}\varphi\in\Sigma$ ($\varphi\in\mathsf{P}\cup\mathsf{N}_1\cup\mathsf{N}_2$).

Lemma 2. Let Σ be π-saturated Λ-consistent set of formulas. The function V^Σ of the Henkin-style Λ-canonical model M^Σ_Λ *is* a valuation function. Moreover, $\leq^\Sigma_{[\mathsf{i}]}$ and $\preccurlyeq^\Sigma_{[\mathsf{i}]}$ are well-defined.

Proof. For any state nominal a, $V^\Sigma(\mathsf{a})$ is shown to be of the form $\{\,|\mathsf{a}|\,\}\times A^\Sigma$ by Proposition 5 (2), the axiom $(\mathtt{Ref})$, Proposition 5 (4) and (5). For any agent nominal i, we use Proposition 5 (3) to establish that $V^\Sigma(\mathsf{i})$ is of the form $W^\Sigma\times\{[\mathsf{i}]\}$. For the well-definedness of $\leq^\Sigma_{[\mathsf{i}]}$, it suffices to obtain:

$$\vdash_\Lambda @_{\mathsf{i}}\mathsf{j}\wedge @_{\mathsf{a}}\mathsf{b}\wedge @_{\mathsf{c}}\mathsf{d}\to(@_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{c}\leftrightarrow @_{\mathsf{j}}@_{\mathsf{b}}\langle\leq\rangle\mathsf{d}),$$

which is Proposition 5 (13). For the well-definedness of $\preccurlyeq^\Sigma_{[\mathsf{i}]}$, the following is sufficient to establish:

$$\vdash_\Lambda @_{\mathsf{i}}\mathsf{i}'\wedge @_{\mathsf{j}}\mathsf{j}'\wedge @_{\mathsf{k}}\mathsf{k}'\to(@_{\mathsf{j}}\langle\sqsubseteq_{\mathsf{i}}\rangle\mathsf{k}\leftrightarrow @_{\mathsf{j}'}\langle\sqsubseteq_{\mathsf{i}'}\rangle\mathsf{k}'),$$

which is also similalry shown to Proposition 5 (13). Finally for the well-definedness of $V^\Sigma(p)$, we can show it by:

$$\vdash_\Lambda @_{\mathsf{i}}\mathsf{j}\wedge @_{\mathsf{a}}\mathsf{b}\to(@_{\mathsf{i}}@_{\mathsf{a}}p\leftrightarrow @_{\mathsf{j}}@_{\mathsf{b}}p),$$

which is shown by Proposition 5 (5). This finishes to prove all the statements in the lemma. □

Lemma 3 (Truth Lemma)**.** Let Σ be π-saturated Λ-consistent set of formulas. For any formula φ, any program π and any pairs $(|\mathsf{a}|,[\mathsf{i}])$, $(|\mathsf{b}|,[\mathsf{j}])$ from M^{Σ},

$$\begin{aligned} M^{\Sigma},(|\mathsf{a}|,[\mathsf{i}]) \Vdash \varphi \quad &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\varphi \in \Sigma; \\ (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\pi}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma. \end{aligned}$$

Proof. We can show these two equivalences by simultaneous induction on a formula φ and a program π. For the cases where φ is a propositional variable, a state-nominal, or an agent-nominal, it is trivial. Moreover, the propositional cases, i.e., the cases where φ is of form $\neg\varphi$ or $\varphi\wedge\psi$ are not difficult to establish. Thus, we focus on the remaining cases here.

1. The case where φ is of the form $\langle\pi\rangle\psi$. We proceed as follows.

$$\begin{aligned} M^{\Sigma},(|\mathsf{a}|,[\mathsf{i}]) \Vdash \langle\pi\rangle\psi \quad &\text{iff} \quad \text{For some } \mathsf{b} \text{ and some } \mathsf{j}\ (((|\mathsf{a}|,[\mathsf{i}])R_{\pi}(|\mathsf{b}|,[\mathsf{j}]) \text{ and } M^{\Sigma},(|\mathsf{b}|,[\mathsf{j}]) \Vdash \psi) \\ &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma \text{ and } @_{\mathsf{b}}@_{\mathsf{j}}\psi \in \Sigma. \end{aligned}$$

The last line implies $@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\psi \in \Sigma$ by Proposition 5 (12). Conversely, we can obtain the last line above from $@_{\mathsf{a}}@_{\mathsf{i}}\langle\pi\rangle\psi \in \Sigma$ by π-saturation.

2. The case where π is an atomic program $\leq$. We proceed as follows:

$$\begin{aligned} (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\leq}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad [\mathsf{i}]=[\mathsf{j}] \text{ and } |\mathsf{a}| \leq^{\Sigma}_{[\mathsf{i}]} |\mathsf{b}| \\ &\text{iff} \quad @_{\mathsf{i}}\mathsf{j} \in \Sigma \text{ and } @_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b} \in \Sigma \\ &\text{iff} \quad @_{\mathsf{i}}\mathsf{j} \wedge @_{\mathsf{i}}@_{\mathsf{a}}\langle\leq\rangle\mathsf{b} \in \Sigma \\ &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle\leq\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma, \end{aligned}$$

where $@_{a}\langle 1_W\rangle\mathsf{b}$ holds by the axiom (1_W) and the last equivalence hold by Proposition 5 (8).

3. The case where π is an atomic program 1_W. We proceed as follows:

$$\begin{aligned} (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{1_W}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad [\mathsf{i}]=[\mathsf{j}] \text{ and } |\mathsf{a}|,|\mathsf{b}| \in W^{\Sigma} \\ &\text{iff} \quad @_{\mathsf{i}}\mathsf{j} \in \Sigma \text{ and } @_{\mathsf{a}}\langle 1_W\rangle\mathsf{b} \in \Sigma \\ &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle 1_W\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma, \end{aligned}$$

where the last equivalence hold by Proposition 5 (9).

4. The case where π is an atomic program $\sqsubseteq_{\mathsf{k}}$. We proceed as follows:

$$\begin{aligned} (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\sqsubseteq_{\mathsf{k}}}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad |\mathsf{a}|=|\mathsf{b}| \text{ and } [\mathsf{i}] \sqsubseteq^{\Sigma}_{[\mathsf{k}]} [\mathsf{j}] \\ &\text{iff} \quad @_{\mathsf{a}}\mathsf{b} \in \Sigma \text{ and } @_{\mathsf{i}}\langle\sqsubseteq_{\mathsf{k}}\rangle\mathsf{j} \in \Sigma \\ &\text{iff} \quad @_{\mathsf{a}}\mathsf{b} \wedge @_{\mathsf{i}}\langle\sqsubseteq_{\mathsf{k}}\rangle@_{\mathsf{j}} \in \Sigma \\ &\text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle\sqsubseteq_{\mathsf{k}}\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma, \end{aligned}$$

where the last equivalence hold by Proposition 5 (10).

5. The case where π is an atomic program 1_A. Similarly proved to the case where π is the atomic program 1_W.

6. The case where π is a compounded program $-\rho$. We proceed as follows:

$$\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{-\rho}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad ((|\mathsf{a}|,[\mathsf{i}]),(|\mathsf{b}|,[\mathsf{j}])) \notin R^{\Sigma}_{\rho} \\
&\text{iff} \quad \neg @_{\mathsf{i}} @_{\mathsf{a}} \langle \rho \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \\
&\text{iff} \quad @_{\mathsf{i}} @_{\mathsf{a}} \neg \langle \rho \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle -\rho \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma,
\end{aligned}$$

where the last equivalence hold due to the axiom $(-)$ of Table 2.

7. The case where π is a compounded program ρ^{-1}. We proceed as follows:

$$\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho^{-1}}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad (|\mathsf{b}|,[\mathsf{j}])R^{\Sigma}_{\rho}(|\mathsf{a}|,[\mathsf{i}]) \\
&\text{iff} \quad @_{\mathsf{b}} @_{\mathsf{j}} \langle \rho \rangle (\mathsf{a} \wedge \mathsf{i}) \in \Sigma \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho^{-1} \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma,
\end{aligned}$$

where the last equivalence hold due to the axiom $(\texttt{Conv})$ of of Table 2.

8. The case where π is a compounded program $\rho_1 \cup \rho_2$.

$$\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_1 \cup \rho_2}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_1}(|\mathsf{b}|,[\mathsf{j}]) \text{ or } (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_2}(|\mathsf{b}|,[\mathsf{j}]) \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \text{ or } @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \text{ by I.H.} \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \rangle (\mathsf{b} \wedge \mathsf{j}) \vee @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \cup \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma,
\end{aligned}$$

where the last equivalence hold by the axiom $(\cup)$ of Table 2.

9. The case where π is a compounded program $\rho_1 \cap \rho_2$.

$$\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_1 \cap \rho_2}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_1}(|\mathsf{b}|,[\mathsf{j}]) \text{ and } (|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho_2}(|\mathsf{b}|,[\mathsf{j}]) \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \text{ and } @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \text{ by I.H.} \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \rangle (\mathsf{b} \wedge \mathsf{j}) \wedge @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho_1 \cap \rho_2 \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma,
\end{aligned}$$

where the last equivalence hold by the axiom $(\cap)$ of Table 2.

10. The case where π is a compounded program $\rho \restriction \mathsf{k}$. We proceed as follows:

$$\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{\rho \restriction \mathsf{k}}(|\mathsf{b}|,[\mathsf{j}]) \quad &\text{iff} \quad (|\mathsf{a}|,[\mathsf{k}])R^{\Sigma}_{\rho}(|\mathsf{b}|,[\mathsf{k}]) \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{k}} \langle \rho \rangle (\mathsf{b} \wedge \mathsf{k}) \in \Sigma \text{ by I.H.} \\
&\text{iff} \quad @_{\mathsf{a}} @_{\mathsf{i}} \langle \rho \restriction \mathsf{k} \rangle (\mathsf{b} \wedge \mathsf{j}) \in \Sigma,
\end{aligned}$$

where the last equivalence hold by the axiom $(\rho \restriction \mathsf{k})$ of Table 2.

11. The case where π is a compounded program $?(\psi_1,\psi_2)$. We proceed as follows:

$$
\begin{aligned}
(|\mathsf{a}|,[\mathsf{i}])R^{\Sigma}_{?(\psi_1,\psi_2)}(|\mathsf{b}|,[\mathsf{j}]) \quad & \text{iff} \quad M^{\Sigma},(|\mathsf{a}|,[\mathsf{i}]) \Vdash \psi_1 \text{ and } M^{\Sigma},(|\mathsf{b}|,[\mathsf{j}]) \Vdash \psi_2 \\
& \text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\psi_1 \in \Sigma \text{ and } @_{\mathsf{b}}@_{\mathsf{j}}\psi_2 \in \Sigma \text{ by I.H.} \\
& \text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\psi_1 \wedge @_{\mathsf{b}}@_{\mathsf{j}}\psi_2 \in \Sigma \text{ by I.H.} \\
& \text{iff} \quad @_{\mathsf{a}}@_{\mathsf{i}}\langle ?(\psi_1,\psi_2)\rangle(\mathsf{b}\wedge\mathsf{j}) \in \Sigma,
\end{aligned}
$$

where the last equivalence hold by the axiom $(?)$ of Table 2. □

The following two previous definitions and lemma enable us to state that our Henkin model satisfies all the required properties for *PR*-model, since all the pairs of our Henkin model is satisfied by some pair of world- and agent-nominals and all axioms for atomis programs of H**PR** of Table 2 does not contain any propositional variables.

Recall from Definition 6 that $M = (F,V)$ is *named* if every pair $(w,i) \in W \times A$ there exists a pair $(\mathsf{a},\mathsf{i}) \in \mathsf{N}_1 \times \mathsf{N}_2$ of nominals such that $w = \underline{\mathsf{a}}$ and $i = \underline{\mathsf{i}}$. Recall also that a formula φ is *pure* if it does not contain any propositional variables. The following lemma is proved similarly to [3, Lemma 7.22].

Lemma 4. Let F be *PR*-frame, V a valuation and φ be a pure formula. If $\varphi\sigma$ is valid in $M = (F,V)$ for every uniform substitution σ, then φ is valid in F.

The following are the statements of Theorem 2, i.e., strong completeness of H**PR**Δ and H$\mathbf{PR}_{(m,n)}\Delta$ by a set Δ of pure formulas:

> Let Δ be a set of pure formulas and H**PR**Δ and H$\mathbf{PR}_{(m,n)}\Delta$ be axiomatic expansions of H**PR** and H$\mathbf{PR}_{(m,n)}$ by Δ, respectively.
>
> 1. If a formula set Σ is consistent in H**PR**Δ then Σ is satisfiable in the class $\mathbb{F}$ defined by Σ of (possibly infinite) *PR*-frames.
> 2. If a formula set Σ is consistent in H$\mathbf{PR}_{(m,n)}\Delta$ then Σ is satisfiable in the class $\mathbb{F}$ defined by Σ of *PR*-frames with fixed m worlds and fixed n agents.

The following provides a proof of Theorem 2.

Proof. We just focus on establishing item 2. Let Λ be H$\mathbf{PR}_{(m.n)}\Delta$. Suppose that Σ is Λ-consistent. By Lemma 1, we can extend Σ to a maximally Λ-consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$ and Σ^+ is named, π-saturated and contains both at least m-distinct states and at least n-distinct agents. Then we define the Henkin-style canonical model $M^{\Sigma^+}_{\Lambda} = (W^{\Sigma^+}, A^{\Sigma^+}, (\leq^{\Sigma^+}_{[\mathsf{i}]}, \preccurlyeq^{\Sigma^+}_{[\mathsf{i}]})_{[\mathsf{i}]\in A^{\Sigma^+}}, V^{\Sigma^+})$. Since $\Sigma \subseteq \Sigma^+$, we obtain $M^{\Sigma^+}, (|\mathsf{a}|,[\mathsf{i}]) \Vdash \varphi$ for all $\varphi \in \Sigma$. Finally, we need to assure that M^{Σ^+} *is* a *PR*-model, i.e., $(\leq^{\Sigma^+}_{[\mathsf{i}]}, \preccurlyeq^{\Sigma^+}_{[\mathsf{i}]})_{[\mathsf{i}]\in A^{\Sigma^+}}$ satisfies the intended properties. Since all axioms for frame properties of Table 2 (except the axiom $(\mathtt{Eq}_{\sqsupseteq})$) do not contain any propositional variable from P (i.e., *pure*) and any substitution instances of them belong to Σ^+, the axioms for frame properties are shown to be valid in the underlying frame F^{Σ^+} of M^{Σ^+} by Lemma 4. This implies that F^{Σ^+} satisfies the intended properties. This finishes to show that all the elements of Σ are simultaneously true at some pair in some *PR*-model. □