

A Modal Logic for the Hide and Seek Game

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Abstract. We study the game of hide and seek in the form of a graph game from a modal logic perspective. A logic is proposed to express moves and strategies in the game, and we show how the addition of an equality constant that models the winning condition of the game makes the logic undecidable. There are certain decidable fragments of first-order logic which behave in a similar fashion with respect to this kind of language extension, and we add a new modal instance to that class. We discuss the relative expressive power of the proposed logic in comparison to the standard modal counterparts and provide a characterization theorem for its expressiveness. We show that the model checking problem for the resulting logic is P-complete. We also study an axiomatization of the logic, and explore the connection with related product logics, which helps us gain more insight towards the subtleties of the proposed framework. Although the logic is designed for the game with two players, a hider and a seeker, our results can easily be transferred to the settings with more players, and this kind of logical exploration enhances our understanding of the game itself.

Keywords: Hide and seek · Cops and robbers · Modal logic · Identity constant · Computational complexity · Product logic · Axiomatization · Expressiveness

1 Introduction

We explore the age-old game of hide and seek from a logic perspective for a formal study of the actions and strategies applied by the players in this game. The game arena is taken to be a graph, and the players move along the edges of the graph. We present a comprehensive survey of the work done in [35, 36, 16]. The game explored is close to that of *cops and robbers*, a popular paradigm studied in the computer science literature [38], among others. Such games played on graphs provide a platform for studying reasoning about social interaction and challenges

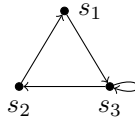
therein arising from the interactions between agents with intertwined objectives. We complement the existing explorations on cops and robbers from algorithmic and combinatorial perspectives (e.g., [38, 14, 30, 21, 39]) with a logical approach, which opens up the possibility of a logical analysis of the game with all its generality. Let us first of all present the basics of the game.

Basics of the hide and seek game. For simplicity, we will only consider the game with two players, Hider (male) and Seeker (female), who are located on a given graph. In each round, Hider and Seeker, in turn, move along a directed edge/arrow starting from their own positions, and if there are no such arrows, they just stay there. The goal of the Seeker is to move to the same position as that of the Hider, while the Hider aims to avoid the Seeker. When the game lasts forever, Hider wins.

The above description still leaves out quite a few details of the game in terms of the information available to the players. There may be different possibilities.

Perfect vs. Imperfect information. Recall the hide and seek game we played when we were children, in which Hider and Seeker usually only have imperfect information, and they may not be aware of the positions of each other. In such a setting, many further aspects need to be specified, for instance, whether players know the game graph or are aware of their own positions [34]. To analyze their strategies in such situations, it is crucial to see how they update their knowledge along the way. For this work, we assume that players have perfect information, in that the whole graph and the players' positions at each stage of the game are available to all the players. For an illustration of the game, let us consider a simple example.

Example 1. Assume that Hider and Seeker are at s_1 and s_2 , respectively, in the graph below.



It is simple to verify that Seeker can win. One of her winning strategies is as follows. In the first round, Hider and Seeker have to move to s_3 and s_1 respectively. Then, in the second round, Hider would not stay at s_3 , as otherwise Seeker will move to the same position immediately, and so he moves to s_2 and Seeker moves to s_3 . Now, what Seeker needs to do is just to wait for Hider at the state s_3 . However, the situation might be very different if we modify the graph: for instance, if we remove the reflexive link from the graph, Hider has a winning strategy in the game.

Several features of the game make it natural to reason about the game with a modal logic approach. One important observation is that we now have two players located at two different nodes. Similar to basic modal logic which considers a state in a Kripke model as an evaluation point for formulas, to model

the independent moves of the two players, we can consider a pair of states as an evaluation point in the model, and consider distinct modalities to express the moves of the players. The evaluation of these two different modalities, one for each player, can then be assessed coordinate-wise with respect to the pair of states. However, to have a desired logical tool, we still need a device to characterize the winning conditions for the players, that is, to model whether the two states of the evaluation pair are the same. To take care of this, a propositional constant expressing the identity relation is added to the existing setting.

Following the ideas mentioned above, we develop a *Logic of Hide and Seek* (LHS) that expresses the game in a natural way. Various properties of the logic are studied in-depth, involving the expressive power, axiomatization, computational behavior and the connections between the logic and the relevant traditions. As we shall see, some findings may look surprising. For instance, although the satisfiability problem for the fragment of LHS without the identity constant is decidable, incorporating the same transforms it into an undecidable one.⁷ Moreover, coming back to the game proper, this kind of logical exploration also reveals some meta-properties of the game: for instance, as we will analyze, the complexity of the model checking problem for our logic provides a proper upper bound for the complexity of solving the game that we consider here. Let us now start the journey.

Outline. In Section 2, we introduce the details of LHS and analyze its applications to the game of hide and seek. Section 3 deals with the expressiveness of the language: by presenting a suitable notion of bisimulation and a first-order translation, we provide a characterization theorem for our logic. In Section 4, we show a complete Hilbert style proof system for a crucial fragment LHS^- of LHS that does not contain the identity constant, and establish a decidability result for it. Section 5 explores the computational behavior of LHS showing that the satisfiability problem for LHS is undecidable, whereas its (finite) model checking problem is P-complete. Section 6 opens up a new direction by embedding LHS into the framework of product logics. Finally, Section 7 provides a discussion on related work, and Section 8 gives pointers to further research.

2 Logic of hide and seek (LHS)

Let us first introduce a logic, LHS, to describe the game of hide and seek, which is followed by some typical validities and an informal discussion about the expressivity of the proposed logic.

⁷ There are other elegant examples that suggest that taking the identity relation into account may change previously decidable logics (without equality) into undecidable ones, e.g., the Gödel class of first-order formulas with identity [26]. A more recent example is the logic of functional dependence [3, 40]. Our logic is one more witness to this phenomenon.

Definition 1. Let $\mathbf{S} = \{p_i^s : i \in \mathbb{N}\}$ and $\mathbf{H} = \{p_i^h : i \in \mathbb{N}\}$ be two disjoint countable sets of propositional variables. The language \mathcal{L} for LHS is given by:

$$\mathcal{L} \ni \varphi ::= p^h \mid p^s \mid \mathcal{I} \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\mathbf{H}]\varphi \mid [\mathbf{S}]\varphi,$$

where $p^h \in \mathbf{H}$ and $p^s \in \mathbf{S}$.

Abbreviations $\top, \perp, \vee, \rightarrow$ are as usual. Intuitively, \mathcal{I} is a constant describing the meeting of the players, and $[\mathbf{H}]$ and $[\mathbf{S}]$ represent the movements of Hider and Seeker respectively. We use $\langle \mathbf{H} \rangle, \langle \mathbf{S} \rangle$ for their dual operators. The notion of subformulas $\text{Sub}(\varphi)$ is defined in a usual manner.

Definition 2. A frame is a tuple $\mathfrak{F} = (W, R)$ such that W is a non-empty set of states (or points) and $R \subseteq W \times W$ is a binary relation on W . A standard relational model $\mathfrak{M} = (W, R, V)$ equips a frame with a valuation function $V : \mathbf{H} \cup \mathbf{S} \rightarrow \mathcal{P}(W)$.⁸ For any $s, t \in W$, we call (\mathfrak{M}, s, t) a pointed LHS-model. For simplicity, we usually write \mathfrak{M}, s, t for it. For each $w \in W$ and $U \subseteq W$, we define $R(w) = \{v \in W : (w, v) \in R\}$ and $R(U) = \bigcup_{u \in U} R(u)$.

Definition 3. Let $\mathfrak{M} = (W, R, V)$ be a model and $s, t \in W$. Truth of formulas $\varphi \in \mathcal{L}$ at (\mathfrak{M}, s, t) , written as $\mathfrak{M}, s, t \models \varphi$, is defined recursively as follows:

$$\begin{aligned} \mathfrak{M}, s, t \models p^h &\Leftrightarrow s \in V(p^h) \\ \mathfrak{M}, s, t \models p^s &\Leftrightarrow t \in V(p^s) \\ \mathfrak{M}, s, t \models \mathcal{I} &\Leftrightarrow s = t \\ \mathfrak{M}, s, t \models \neg\varphi &\Leftrightarrow \mathfrak{M}, s, t \not\models \varphi \\ \mathfrak{M}, s, t \models \varphi \wedge \psi &\Leftrightarrow \mathfrak{M}, s, t \models \varphi \text{ and } \mathfrak{M}, s, t \models \psi \\ \mathfrak{M}, s, t \models [\mathbf{H}]\varphi &\Leftrightarrow \mathfrak{M}, s', t \models \varphi \text{ for all } s' \in R(s) \\ \mathfrak{M}, s, t \models [\mathbf{S}]\varphi &\Leftrightarrow \mathfrak{M}, s, t' \models \varphi \text{ for all } t' \in R(t) \end{aligned}$$

The set $\llbracket \varphi \rrbracket^{\mathfrak{M}} := \{(s, t) : \mathfrak{M}, s, t \models \varphi\}$ is said to be the truth set of formula φ in \mathfrak{M} . Also, notions such as *satisfiability*, *validity* and *modal equivalence* can be given in a usual way. Let us denote LHS^- to be the fragment of LHS without the constant \mathcal{I} . In what follows, we use \mathcal{L}^- for the part of \mathcal{L} without \mathcal{I} , which is the language for LHS^- .

Remark 1. The design of LHS^- is close to the product logic $\mathbf{K} \times \mathbf{K}$ [22], while LHS is close to $\mathbf{K} \times^\delta \mathbf{K}$ [29, 31, 28] that extends $\mathbf{K} \times \mathbf{K}$ with a *diagonal constant* δ .⁹ But there are also crucial differences between our frameworks and the tradition of many-dimensional logics, and we defer the discussion to Sections 4, 6 and 7.

Definition 4. Let $\mathfrak{M} = (W, R, V)$ be a model and $U \subseteq W$. We say a model $\mathfrak{M}' = (W', R', V')$ is generated from $\mathfrak{M} = (W, R, V)$ by U , if \mathfrak{M}' is the smallest model satisfying the following: $U \subseteq W'$, $R(W') \subseteq W'$, $R' = R \cap (W' \times W')$, and for each $p \in \mathbf{H} \cup \mathbf{S}$, $V'(p) = V(p) \cap W'$.

⁸ For any set A , we use $\mathcal{P}(A)$ for its power set.

⁹ In a product model, δ holds at a state (s, t) just in the case that $s = t$.

By induction on formulas, it can be easily proved that:

Proposition 1. *Let $\mathfrak{M}' = (W', R', V')$ be a submodel of $\mathfrak{M} = (W, R, V)$ generated by $\{s, t\} \subseteq W$. For any formula $\varphi \in \mathcal{L}$, $\mathfrak{M}, s, t \models \varphi$ iff $\mathfrak{M}', s, t \models \varphi$.*

Here are some principles that are useful to see the basic features of logic. In what follows, the notation \star refers to either H or S. First, we have $[\star](\varphi \rightarrow \psi) \rightarrow ([\star]\varphi \rightarrow [\star]\psi)$, saying that both [H] and [S] can be distributed over an implication. What is more interesting is the interaction between different Boolean connectives w.r.t. the constant \mathcal{I} , e.g., $\langle \star \rangle (\mathcal{I} \wedge \varphi) \rightarrow [\star] (\mathcal{I} \rightarrow \varphi)$ that captures the uniqueness of the state that is identical to a given one. Moreover, we have $\mathcal{I} \rightarrow (\langle H \rangle \top \leftrightarrow \langle S \rangle \top)$, stating the ‘symmetry’ of the structure w.r.t. \mathcal{I} -pairs, and $\mathcal{I} \rightarrow ([S]\langle H \rangle \mathcal{I} \wedge [H]\langle S \rangle \mathcal{I})$, expressing a kind of ‘closure’ property. Finally, validities of LHS are *not* closed under substitution, say, both the validities $p^s \rightarrow [H]p^s$ and $p^h \rightarrow [S]p^h$ may fail after replacing p^s/p^h with some p^h/p^s .

Going back to the game of hide and seek itself, local one-step winning positions for each player can be expressed in our language as follows:

$$\text{Hider: } \langle H \rangle [S] \neg \mathcal{I} \quad \text{Seeker: } [H] \langle S \rangle \mathcal{I}$$

More generally, winning positions for Hider and Seeker can be described as:

$$\text{Hider: } \forall n (\langle H \rangle [S])^n \neg \mathcal{I} \quad \text{Seeker: } \exists n ([H] \langle S \rangle)^n \mathcal{I}$$

The interactions $\langle H \rangle [S] / [H] \langle S \rangle$ are expressed with two separate modalities, but they are considered as a single unit. These are not expressible in our language due to their infinite length, but we can overcome this with the idea of modal substitutions [46, 9] that provide a finitary way to express such countable Boolean operations.

Remark 2. There are other ways to give suitable logics capturing the game. For instance, one can replace \mathcal{I} with a new constant C , denoting ‘catching’: $\mathfrak{M}, s, t \models C$ iff $R(s) \subseteq R(t)$, which describes that all states accessible to Hider are accessible to Seeker as well. In contrast to \mathcal{I} which states that Seeker has already won, C indicates that she can win in the next round. They amount to the same condition for games of perfect information: if Seeker has the ability to meet Hider she will actually do that, if she is rational. However, from a logical perspective, their interpretations are entirely different, leading to distinct expressive features. The constant C is also useful in describing Seeker-win graphs in the game [38], and we leave it for future work.

3 Expressiveness

To have a better understanding of our logic, in this section we explore the expressiveness of the framework. The notion of bisimulation is an important tool for this, and we will explore a suitable notion tailored to our logic. Also, with the help of a first-order translation, we will prove a characterization theorem for LHS. The techniques used for these can be easily adapted to establish the corresponding results for LHS^- .

3.1 Bisimulations

To have a desired notion of bisimulation, we take the standard bisimulation (see e.g., [12]) as the benchmark and investigate the relations between expressiveness of basic modal logic K_{\square} , LHS^{-} and LHS . Let us start by comparing that for LHS^{-} and K_{\square} .

The standard bisimulation, denoted by \leftrightarrow^s , provides us a semantic characterization of the expressiveness of the basic modal language \mathcal{L}_{\square} . And at a first glance, the semantic design of LHS^{-} is similar to that of the basic modal logic, except that we now need to consider two states simultaneously when evaluating formulas. So, is logic LHS^{-} invariant under the standard notion? First, we provide a positive answer in the following sense:

Proposition 2. *If $(\mathfrak{M}, w) \leftrightarrow^s (\mathfrak{M}', w')$ and $(\mathfrak{M}, v) \leftrightarrow^s (\mathfrak{M}', v')$, then (\mathfrak{M}, w, v) and (\mathfrak{M}', w', v') satisfy the same formulas of LHS^{-} .*

Proof. The proof is by induction on LHS^{-} -formulas φ . We omit the cases for Boolean connectives \neg and \wedge , and only consider p^h and $[S]\psi$, since p^s and $[H]\psi$ can be proved similarly.

(1). First, formula φ is p^h . The following sequence of equivalences holds:

$$\mathfrak{M}, w, v \models p^h \text{ iff } w \in V(p^h) \text{ iff } w' \in V'(p^h) \text{ iff } \mathfrak{M}', w', v' \models p^h$$

The second equivalence holds by the facts that $(\mathfrak{M}, w) \leftrightarrow^s (\mathfrak{M}', w')$ and that truth of basic modal formula is invariant under standard bisimulation [12].

(2). φ is $[S]\psi$. It suffices to show that $\mathfrak{M}, w, v \not\models \varphi$ implies $\mathfrak{M}', w', v' \not\models \varphi$. Suppose $\mathfrak{M}, w, v \not\models \varphi$. Then there is some $s \in R(v)$ with $\mathfrak{M}, w, s \not\models \psi$. From $(\mathfrak{M}, v) \leftrightarrow^s (\mathfrak{M}', v')$, it follows that there exists $s' \in R'(v')$ with $(\mathfrak{M}, s) \leftrightarrow^s (\mathfrak{M}', s')$. By the inductive hypothesis, $\mathfrak{M}', w', s' \not\models \psi$, which entails $\mathfrak{M}', w', v' \not\models [S]\psi$. \square

Therefore, the standard bisimulation is strong enough to measure the expressive power of LHS^{-} . However, we also have a negative result as the following:

Proposition 3. *There are (\mathfrak{M}, w, v) and (\mathfrak{M}', w', v') s.t. they satisfy the same LHS^{-} -formulas but one of $(\mathfrak{M}, w) \leftrightarrow^s (\mathfrak{M}', w')$, $(\mathfrak{M}, v) \leftrightarrow^s (\mathfrak{M}', v')$ may not hold.¹⁰*

Proof. Consider \mathfrak{M} and \mathfrak{M}' depicted in Figure 1. It holds that (\mathfrak{M}, w_1, w_2) and $(\mathfrak{M}', v_1, v_2)$ satisfy the same \mathcal{L} -formulas, and hence, the same \mathcal{L}^{-} -formulas, but we do not have $(\mathfrak{M}, w_1) \leftrightarrow^s (\mathfrak{M}', v_1)$. \square

Intuitively, the failure originates from the ‘evaluation-gap’ between the two states in our pointed LHS -models (\mathfrak{M}, s, t) : when considering atomic properties of s , both LHS^{-} and LHS can only describe those in \mathbf{H} , but not the ones in \mathbf{S} . Let us now introduce a notion of bisimulation for LHS :

¹⁰ Strictly speaking, a negative result holds even for the basic modal logic (see [12]). However, it is still ideal if the notion of bisimulation can behave well in a large class of models (e.g., *image-finite models*). But, as illustrated by the counterexample used to show the result, the standard notion even excludes situations that are very simple but cannot be distinguished by LHS^{-} .



Fig. 1. Two pointed LHS-models (\mathfrak{M}, w_1, w_2) and $(\mathfrak{M}', v_1, v_2)$ satisfying the same \mathcal{L} -formulas.

Definition 5. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be models. We say that a non-empty relation $Z \subseteq (W \times W) \times (W' \times W')$ is an LHS-bisimulation between \mathfrak{M} and \mathfrak{M}' , notation $Z : \mathfrak{M} \Leftrightarrow \mathfrak{M}'$, if the following conditions hold for all $s, t, v \in W$ and $s', t', v' \in W'$:

Atom: If $(s, t)Z(s', t')$, then for all $p \in \mathbf{H} \cup \mathbf{S}$, $\mathfrak{M}, s, t \models p$ iff $\mathfrak{M}', s', t' \models p$.

Meet: If $(s, t)Z(s', t')$, then $s = t$ if and only if $s' = t'$.

Zig_H: If $(s, t)Z(s', t')$ and $v \in R(s)$, then there is $v' \in R'(s')$ s.t. $(v, t)Z(v', t')$.

Zig_S: If $(s, t)Z(s', t')$ and $v \in R(t)$, then there is $v' \in R'(t')$ s.t. $(s, v)Z(s', v')$.

Zag_H: If $(s, t)Z(s', t')$ and $v' \in R'(s')$, then there is $v \in R(s)$ s.t. $(v, t)Z(v', t')$.

Zag_S: If $(s, t)Z(s', t')$ and $v' \in R'(t')$, then there is $v \in R(t)$ s.t. $(s, v)Z(s', v')$.

If there is an LHS-bisimulation linking (s, t) and (s', t') , then we say that (\mathfrak{M}, s, t) is LHS-bisimilar to (\mathfrak{M}', s', t') and write $(\mathfrak{M}, s, t) \Leftrightarrow (\mathfrak{M}', s', t')$. By dropping the clause **Meet**, we get the notion for LHS⁻, and when (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') are LHS⁻-bisimilar, we write $(\mathfrak{M}, s, t) \Leftrightarrow^- (\mathfrak{M}', s', t')$.

With this definition, it is now easy to check that (\mathfrak{M}, w_1, w_2) and $(\mathfrak{M}', v_1, v_2)$ in Figure 1 are bisimilar. Although the clauses above look rather routine, it is instructive to notice some subtle aspects of the definition that are in line with our previous observation: the condition **Atom** in effect just requires that $V(s) \cap \mathbf{H} = V'(s') \cap \mathbf{H}$ and $V(t) \cap \mathbf{S} = V'(t') \cap \mathbf{S}$, but s and s' may satisfy different properties, say, from \mathbf{S} , and t and t' may satisfy different properties, say, from \mathbf{H} . Moreover, the clause **Meet** aims to deal with the constant \mathcal{I} , and the others are analogous to the zigzag conditions in standard situations. With Definition 5, it holds that:

Proposition 4. If $(\mathfrak{M}, s, t) \Leftrightarrow (\mathfrak{M}', s', t')$, then (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') satisfy the same LHS-formulas. Also, if $(\mathfrak{M}, s, t) \Leftrightarrow^- (\mathfrak{M}', s', t')$, then they satisfy the same LHS⁻-formulas.

Proof. The proof proceeds by induction on LHS-formulas and LHS⁻-formulas.

For each model $\mathfrak{M} = (W, R, V)$ and $X \subseteq W \times W$, a set Γ of formulas is said to be *finitely satisfiable in X*, if for each finite subset Δ of Γ , there exists $\langle s, t \rangle \in X$ such that $\mathfrak{M}, s, t \models \Delta$. Now we will show a result in the converse direction, by focusing on a class of LHS-saturated models:

Definition 6. A model $\mathfrak{M} = (W, R, V)$ is said to be LHS-saturated, if for any set Φ of formulas and states $w, v \in W$, it holds that:

- If Φ is finitely satisfiable in $R(w) \times \{v\}$, then the whole set Φ is satisfiable in $R(w) \times \{v\}$, and

- If Φ is finitely satisfiable in $\{w\} \times R(v)$, then the whole set Φ is satisfiable in $\{w\} \times R(v)$.

One can check that any finite model is LHS-saturated. Moreover, in terms of infinite \mathfrak{M} , it intuitively requires that \mathfrak{M} contains ‘enough’ states: for instance, if every finite subset of Φ can be satisfied by some pairs in $R(w) \times \{v\}$, then there must be a pair satisfying Φ itself. By restricting Φ to the fragment without \mathcal{I} , we have a notion for LHS^- , called LHS^- -saturation.

Proposition 5. *For all \mathfrak{M} and \mathfrak{M}' that are LHS-saturated, if (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') satisfy the same formulas of LHS, then $(\mathfrak{M}, s, t) \Leftrightarrow (\mathfrak{M}', s', t')$. Moreover, when \mathfrak{M} and \mathfrak{M}' are LHS^- -saturated, if (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') satisfy the same formulas of LHS^- , then $(\mathfrak{M}, s, t) \Leftrightarrow^- (\mathfrak{M}', s', t')$.*

Proof. Assume that \mathfrak{M} and \mathfrak{M}' are LHS-saturated, and (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') satisfy the same formulas of LHS. We show that the modal equivalence relation itself is a bisimulation relation for LHS. Clause **Atom** holds immediately by the fact that (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') are modally equivalent w.r.t. LHS. In what follows, we just show **Meet** and **Zig_H** are satisfied.

(1). **Meet.** We have the following equivalences: $s = t$ iff $\mathfrak{M}, s, t \models \mathcal{I}$ iff $\mathfrak{M}', s', t' \models \mathcal{I}$ iff $s' = t'$. So, the condition is satisfied.

(2). **Zig_H.** Assume that $u \in W$ with Rsu . Let Φ denote the set of LHS-formulas that are true at \mathfrak{M}, u, t . For each finite $\Gamma \subseteq \Phi$, $\mathfrak{M}, s, t \models \langle H \rangle \wedge \Gamma$. Thus, $\mathfrak{M}, s', t' \models \langle H \rangle \wedge \Gamma$. So, Φ is finitely satisfiable w.r.t. $R'(s') \times \{t'\}$. Since \mathfrak{M}' is LHS-saturated, there exists $u' \in W'$ such that $R's'u'$ and each formula of Φ is true at \mathfrak{M}', u', t' . Thus, (\mathfrak{M}, u, t) and (\mathfrak{M}', u', t') satisfy the same LHS-formulas.

So, it concludes that $(\mathfrak{M}, s, t) \Leftrightarrow (\mathfrak{M}', s', t')$. Moreover, by dropping the consideration on \mathcal{I} , we can prove the case involving LHS^- . \square

Therefore, w.r.t. LHS-/ LHS^- -saturated models, our notion of bisimulation coincides with the corresponding notion of modal equivalence. Now, we end this part with the following result concerning the relations among aforementioned varieties of bisimulations:

Proposition 6. *W.r.t. the three kinds of bisimulations \Leftrightarrow^s , \Leftrightarrow and \Leftrightarrow^- , we have the following:*

- (1). Both \Leftrightarrow^s and \Leftrightarrow are strictly stronger than \Leftrightarrow^- :
 - (1.1). If $(\mathfrak{M}, w) \Leftrightarrow^s (\mathfrak{M}', w')$ and $(\mathfrak{M}, v) \Leftrightarrow^s (\mathfrak{M}', v')$, then $(\mathfrak{M}, w, v) \Leftrightarrow^- (\mathfrak{M}', w', v')$. But the converse does not always hold.
 - (1.2). If $(\mathfrak{M}, w, v) \Leftrightarrow (\mathfrak{M}', w', v')$, then $(\mathfrak{M}, w, v) \Leftrightarrow^- (\mathfrak{M}', w', v')$. But the converse does not always hold.
- (2). \Leftrightarrow^s and \Leftrightarrow are incomparable:
 - (2.1). When $(\mathfrak{M}, w) \Leftrightarrow^s (\mathfrak{M}', w')$ and $(\mathfrak{M}, v) \Leftrightarrow^s (\mathfrak{M}', v')$, $(\mathfrak{M}, w, v) \Leftrightarrow (\mathfrak{M}', w', v')$ can fail.
 - (2.2). When $(\mathfrak{M}, w, v) \Leftrightarrow (\mathfrak{M}', w', v')$, $(\mathfrak{M}, w) \Leftrightarrow^s (\mathfrak{M}', w')$ and $(\mathfrak{M}, v) \Leftrightarrow^s (\mathfrak{M}', v')$ can fail.

Proof. We show the two claims one by one.

(1). The relation between \leftrightarrow^s and \leftrightarrow^- follows from Propositions 2, 3 and 5. Also, \leftrightarrow is stronger than \leftrightarrow^- . For an example, consider the models in Figure 2: $(\mathfrak{M}, w_1, w_1) \leftrightarrow^- (\mathfrak{M}', v_1, v_1)$, but $\mathfrak{M}, w_1, w_1 \models \langle H \rangle \langle S \rangle \neg \mathcal{I}$ and $\mathfrak{M}', v_1, v_1 \not\models \langle H \rangle \langle S \rangle \neg \mathcal{I}$. Now, by Proposition 4, we do not have $(\mathfrak{M}, w_1, w_1) \leftrightarrow (\mathfrak{M}', v_1, v_1)$.

(2). In the models in Figure 2, the states w_1 and v_1 cannot be distinguished by the basic modal language, but this would not be the case when we consider LHS. Thus, standard bisimulations need not be bisimulations of LHS. On the other hand, using the models in Figure 1, it is not hard to see that bisimulations of LHS may also be excluded by the notion of standard bisimulation. \square

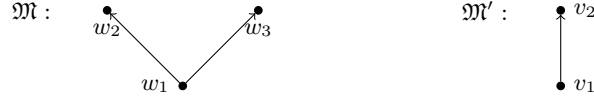


Fig. 2. $(\mathfrak{M}, w_1, w_1) \leftrightarrow^- (\mathfrak{M}', v_1, v_1)$, but not $(\mathfrak{M}, w_1, w_1) \leftrightarrow (\mathfrak{M}', v_1, v_1)$.

Properties of LHS- and LHS⁻-bisimulation explored here are very basic, and several further questions are worth studying. For instance,

Open problem. What is the computational complexity of checking for bisimulation of LHS or LHS⁻? Are they as complex as each other?

3.2 Characterization theorem

In this part, we shall see that as the case of the standard modal logic, LHS can also be embedded into FOL. With the help of this, we will develop a van Benthem style characterization theorem for the logic, which identifies the counterpart of LHS in FOL. Let \mathcal{L}^1 be the first-order language consisting of a countable set $\mathbf{P} = \{P_i^H, P_i^S : i \in \mathbb{N}\}$ of unary predicates, a binary relation R and equality \equiv .

Definition 7. For any two variables x and y , the first-order translation $T_{(x,y)} : \mathcal{L} \rightarrow \mathcal{L}^1$ for LHS is given recursively as follows:

$$\begin{aligned} T_{(x,y)}(p_i^h) &:= P_i^H x & T_{(x,y)}(p_i^s) &:= P_i^S y & T_{(x,y)}(\mathcal{I}) &:= (x \equiv y) \\ T_{(x,y)}(\neg \varphi) &:= \neg T_{(x,y)}(\varphi) & T_{(x,y)}(\varphi \wedge \psi) &:= T_{(x,y)}(\varphi) \wedge T_{(x,y)}(\psi) \\ T_{(x,y)}([\mathbf{H}]\varphi) &:= \forall z (R x z \rightarrow T_{(z,y)}(\varphi)) & T_{(x,y)}([\mathbf{S}]\varphi) &:= \forall z (R y z \rightarrow T_{(x,z)}(\varphi)) \end{aligned}$$

For any set $\Phi \subseteq \mathcal{L}$ of formulas, we define $T_{(x,y)}(\Phi) := \{T_{(x,y)}(\varphi) : \varphi \in \Phi\}$.

Now, by induction on $\varphi \in \mathcal{L}$, we can show the correctness of the translation:

Proposition 7. Let $\mathfrak{M} = (W, R, V)$ be a model and $s, t \in W$. Then for all variables x, y and $\varphi \in \mathcal{L}$, $\mathfrak{M}, s, t \models \varphi$ iff $\mathfrak{M} \models T_{(x,y)}(\varphi)[s, t]$.¹¹

¹¹ By $\mathfrak{M} \models T_{(x,y)}(\varphi)[s, t]$, we mean that when values of x, y are s, t respectively, $T_{(x,y)}(\varphi)$ is satisfied by \mathfrak{M} .

Also, it is simple to see from the translation that every LHS-formula can be translated to an \mathcal{L}^1 -formula with at most 3 variables.

Corollary 1. *LHS can be reduced into the 3 variable fragment of FOL, with a function having a polynomial size increase.*

A model $\mathfrak{M} = (W, R, V)$ realizes $\Gamma(x_1, \dots, x_n) \subseteq \mathcal{L}^1$ if there are $a_1, \dots, a_n \in W$ s.t. for all $\gamma \in \Gamma$, $\mathfrak{M} \models \gamma[a_1, \dots, a_n]$. Let $A \subseteq W$. For each $a \in A$, let o_a be a constant symbol. Let $\mathcal{L}_A^1 = \mathcal{L}^1 \cup \{o_a : a \in A\}$ and let \mathfrak{M}_A denote the \mathcal{L}_A^1 -expansion of \mathfrak{M} s.t. for all $a \in A$, o_a has the value a .

Definition 8. *A model $\mathfrak{M} = (W, R, V)$ is ω -saturated, if for all finite $A \subseteq W$, \mathfrak{M}_A realizes every $\Gamma(x) \subseteq \mathcal{L}_A^1$ whose finite subsets are all realized in \mathfrak{M}_A .*

Proposition 8. *All ω -saturated models $\mathfrak{M} = (W, R, V)$ are LHS-saturated.*

Proof. Let $\Sigma \subseteq \mathcal{L}$ be finitely satisfiable in $R(w) \times \{v\}$ and $w, v \in W$. Then let $\Delta(x) = \{Ro_w x\} \cup \{T_{(x,y)}(\varphi)[y/o_v] : \varphi \in \Sigma\}$. Every finite subset of $\Delta(x)$ is realized by some $u \in R(w)$ in $\mathfrak{M}_{\{w,v\}}$ (Proposition 7). Since \mathfrak{M} is ω -saturated, $\Delta(x)$ is realized in $\mathfrak{M}_{\{w,v\}}$. So, there is some $u \in W$ such that $\mathfrak{M}_{\{w,v\}} \models \Delta(x)[u]$. Thus $u \in R(w)$ and $\mathfrak{M}, u, v \models \Sigma$. Similarly, if Σ is finitely satisfiable in $\{w\} \times R(v)$, then Σ is satisfiable in $\{w\} \times R(v)$. \square

Corollary 2. *Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be ω -saturated models, $s, t \in W$ and $s', t' \in W'$. If (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') satisfy the same \mathcal{L} -formulas, then $(\mathfrak{M}, s, t) \Leftrightarrow (\mathfrak{M}', s', t')$.*

Let \mathfrak{M} be a model, \mathbb{I} a countable set and U an incomplete ultrafilter over \mathbb{I} . Then we write $\prod_U \mathfrak{M}$ for the ultrapower of \mathfrak{M} modulo U .¹²

Proposition 9. *Let $\mathfrak{M} = (W, R, V)$ be a model, \mathbb{I} a countable set and U an incomplete ultrafilter over \mathbb{I} . For each $w \in W$, let $f_w = \mathbb{I} \times \{w\}$. Then,*

1. $\prod_U \mathfrak{M}$ is ω -saturated.
2. For any $\alpha(x, y) \in \mathcal{L}^1$ and $s, t \in W$, $\mathfrak{M} \models \alpha[s, t]$ iff $\prod_U \mathfrak{M} \models \alpha[(f_s)_U, (f_t)_U]$.
3. For any \mathcal{L} -formula φ and $s, t \in W$, $\mathfrak{M}, s, t \models \varphi$ iff $\prod_U \mathfrak{M}, (f_s)_U, (f_t)_U \models \varphi$.

Proof. The first item follows from [15, p.384, Theorem 6.1.1]. The second follows from [15, p.217, Theorem 4.1.9]. The last one follows from the second item and Proposition 7 immediately. \square

We say that an \mathcal{L}^1 -formula $\alpha(x, y)$ is invariant for LHS-bisimulation, if for all (\mathfrak{M}, s, t) and (\mathfrak{M}', s', t') that are LHS-bisimilar, $\mathfrak{M} \models \alpha[s, t]$ iff $\mathfrak{M}' \models \alpha[s', t']$. Now we can provide the following:

Theorem 1. *For any $\alpha(x, y) \in \mathcal{L}^1$, $\alpha(x, y)$ is invariant for LHS-bisimulation if and only if $\models \alpha \leftrightarrow \beta$ for some $\beta(x, y) \in T_{(x,y)}(\mathcal{L})$.*

¹² For the definitions of *ultrafilter* and *ultrapower of models*, see [12, pp.491-493, Definition A.12 and Definition A.18].

Proof. The right-to-left direction holds directly by Proposition 4. For the other direction, let $\alpha(x, y) \in \mathcal{L}^1$ be invariant for LHS-bisimulation. Define $\text{modal}(\alpha) := \{\beta \in \mathbb{T}_{(x,y)}(\mathcal{L}) : \alpha \models \beta\}$. We are going to show that $\text{modal}(\alpha) \models \alpha$. Let \mathfrak{M} be a model with $\mathfrak{M} \models \text{modal}(\alpha)[a, b]$. We now prove that $\mathfrak{M} \models \alpha[a, b]$. Let Φ be a set of formulas defined by

$$\Phi := \{\beta(x, y) \in \mathbb{T}_{(x,y)}(\mathcal{L}) : \mathfrak{M} \models \beta[a, b]\} \cup \{\alpha(x, y)\}.$$

We claim that Φ is satisfiable. Suppose not. Then there is a finite $\Phi_0 \subseteq \Phi$ with $\Phi_0 \models \neg\alpha$, which entails $\alpha \models \neg \bigwedge \Phi_0$ and so $\mathfrak{M} \models \neg \bigwedge \Phi_0[a, b]$. Note that $\bigwedge \Phi_0 \in \Phi$, we see $\mathfrak{M} \models \bigwedge \Phi_0[a, b]$, which is a contradiction. Thus, Φ is satisfiable and there is a model \mathfrak{N} and states w, u with $\mathfrak{N} \models \Phi[w, u]$. Then by Proposition 7, (\mathfrak{M}, a, b) and (\mathfrak{N}, w, u) satisfy the same \mathcal{L} -formulas. Let U be an incomplete ultrafilter over \mathbb{N} . Then by Proposition 9(3), $(\prod_U \mathfrak{M}, (f_a)_U, (f_b)_U)$ and $(\prod_U \mathfrak{N}, (f_c)_U, (f_d)_U)$ satisfy the same \mathcal{L} -formulas. By Proposition 9(1) and Corollary 2, $(\prod_U \mathfrak{M}, (f_a)_U, (f_b)_U) \Leftrightarrow (\prod_U \mathfrak{N}, (f_w)_U, (f_u)_U)$. Since $\mathfrak{N} \models \alpha[w, u]$, by Proposition 9(2), $\prod_U \mathfrak{N} \models \alpha[(f_w)_U, (f_u)_U]$. Since $\alpha(x, y)$ is invariant for LHS-bisimulation, we have $\prod_U \mathfrak{M} \models \alpha[(f_a)_U, (f_b)_U]$. By Proposition 9(2), $\mathfrak{M} \models \alpha[a, b]$. Hence, $\text{modal}(\alpha) \models \alpha$. By the Compactness Theorem, there is a finite $\Sigma \subseteq \text{modal}(\alpha)$ such that $\Sigma \models \alpha$. Then we see that $\models \alpha \leftrightarrow \bigwedge \Sigma$. \square

It is worth noting that by adapting the arguments for LHS, we obtain a characterization theorem for LHS^- . We leave the details to the readers.¹³

As we have seen from this section, many properties of LHS^- can be achieved by simply adjusting the techniques developed for LHS. Given this and the simple-looking of \mathcal{I} , one may guess that \mathcal{I} does not have significant impacts on the behavior of LHS. However, in the next two sections, we will establish a series of results indicating that \mathcal{I} changes LHS drastically: for instance, our results in Section 4 will indicate that LHS^- is decidable, while Section 5 will show that LHS is undecidable.

4 Axiomatization and decidability of LHS^-

In this section, we will provide a proof system for LHS^- , with which we will also show that its satisfiability problem is decidable. To achieve the former, we will first separate \mathcal{L}^- into two parts, the ‘Seeker part’ and the ‘Hider part’, and then build a desired calculus base on the separation. As we shall see, containing two kinds of propositional variables in \mathcal{L}^- makes LHS^- very different from its counterpart $\mathbb{K} \times \mathbb{K}$ in product logic. Let us now introduce the details.

¹³ Atomic logics and the van Benthem characterization theorem for them are studied in [2]. As pointed out by Guillaume Aucher, LHS^- is an atomic logic and thus the characterization theorem follows immediately. However, the case for LHS is slightly different: the operator \mathcal{I} is a constant, which means that LHS is not an atomic logic. One might try to generalize the results in [2] and obtain more on expressive power of those logics.

A formula $\varphi \in \mathcal{L}^-$ is *clean* if it contains only \mathbf{H} -modality or \mathbf{S} -modality (but not both). Formulas in the language \mathcal{L}^- of \mathbf{LHS}^- may contain nested \mathbf{H} -modality and \mathbf{S} -modality. However, as we shall see, *every $\varphi \in \mathcal{L}^-$ is logically equivalent to a Boolean combination of some clean formulas.*

Definition 9. Languages $\mathcal{L}_{\mathbf{H}}$ and $\mathcal{L}_{\mathbf{S}}$ are given by:

$$\begin{aligned}\mathcal{L}_{\mathbf{H}} \ni \varphi &::= p^h \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid [\mathbf{H}]\varphi, \\ \mathcal{L}_{\mathbf{S}} \ni \varphi &::= p^s \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid [\mathbf{S}]\varphi,\end{aligned}$$

where $p^h \in \mathbf{H}$ and $p^s \in \mathbf{S}$.

Let $\mathbf{K}_{\mathbf{H}}$ and $\mathbf{K}_{\mathbf{S}}$ denote *the minimal normal modal logics with the languages $\mathcal{L}_{\mathbf{H}}$ and $\mathcal{L}_{\mathbf{S}}$, respectively.* As the case for \mathbf{K}_{\square} , the satisfiability problems for both the logics are decidable (cf. [5]). Also, except for the difference of the languages, their proof systems are the same as that for \mathbf{K}_{\square} [12], and we write $\mathbf{K}_{\mathbf{H}}$ and $\mathbf{K}_{\mathbf{S}}$ for them. In what follows, we write \models_1 for *the usual one-dimensional satisfaction relation.* By induction on formulas, we can show that:

Proposition 10. *Let $\mathfrak{M} = (W, R, V)$ be a $\mathbf{K}_{\mathbf{H}}$ -model and $\mathfrak{N} = (U, S, V')$ a $\mathbf{K}_{\mathbf{S}}$ -model where $W \cap U = \emptyset$. Let $\psi \in \mathcal{L}_{\mathbf{H}}$, $\gamma \in \mathcal{L}_{\mathbf{S}}$. Then for all $s \in W$ and $t \in U$,*

- (1). $\mathfrak{M}, s \models_1 \psi$ if and only if $\mathfrak{M} \uplus \mathfrak{N}, s, t \models \psi$, and
- (2). $\mathfrak{N}, t \models_1 \gamma$ if and only if $\mathfrak{M} \uplus \mathfrak{N}, s, t \models \gamma$,

where $\mathfrak{M} \uplus \mathfrak{N}$ is the \mathbf{LHS} -model defined by $\mathfrak{M} \uplus \mathfrak{N} = (W \cup U, R \cup S, V \cup V')$. The \mathbf{LHS} -model $\mathfrak{M} \uplus \mathfrak{N}$ is called *the disjoint union of \mathfrak{M} and \mathfrak{N} .*

Let \mathfrak{M} be a $\mathbf{K}_{\mathbf{H}}$ -model and \mathfrak{N} a $\mathbf{K}_{\mathbf{S}}$ -model. Since there are always isomorphic copies of them with disjoint domains, we can always assume that the domains of \mathfrak{M} and \mathfrak{N} are disjoint and construct the disjoint union of \mathfrak{M} and \mathfrak{N} .

Given an \mathbf{LHS} -model $\mathfrak{M} = (W, R, V)$, by restricting V to \mathbf{H} (we write $V|_{\mathbf{H}}$ for it), we can obtain a model $\mathfrak{M}|_{\mathbf{H}} = (W, R, V|_{\mathbf{H}})$ for $\mathcal{L}_{\mathbf{H}}$, and similarly, by restricting V to \mathbf{S} (we write $V|_{\mathbf{S}}$ for it), we can get a model $\mathfrak{M}|_{\mathbf{S}} = (W, R, V|_{\mathbf{S}})$ for $\mathcal{L}_{\mathbf{S}}$. By induction on formulas, it is simple to prove that:

Proposition 11. *Let (\mathfrak{M}, s, s) be a pointed \mathbf{LHS} -model.*

- (1). For any $\varphi \in \mathcal{L}_{\mathbf{H}}$, $\mathfrak{M}, s, s \models \varphi$ iff $\mathfrak{M}|_{\mathbf{H}}, s \models_1 \varphi$.
- (2). For any $\varphi \in \mathcal{L}_{\mathbf{S}}$, $\mathfrak{M}, s, s \models \varphi$ iff $\mathfrak{M}|_{\mathbf{S}}, s \models_1 \varphi$.

Before the next step, let us first recall some concepts and facts about propositional logic. Let \mathcal{L}_p denote the propositional language whose propositional variables come from $\mathbf{H} \cup \mathbf{S}$. For each $\varphi \in \mathcal{L}_p$, we write $\varphi(p_1, \dots, p_n)$ if the propositional variables occurring in φ are among p_1, \dots, p_n . Let $\varphi(\alpha_1, \dots, \alpha_n)$ denote the formula obtained from $\varphi(p_1, \dots, p_n)$ by simultaneously substituting p_1, \dots, p_n with $\alpha_1, \dots, \alpha_n$ respectively. Let \mathbf{PL} denote the set of all valid formulas in \mathcal{L}_p . A sound and complete Hilbert style calculus \mathbf{PL} for \mathbf{PL} can be given in a usual way.

Definition 10. A formula $\varphi \in \mathcal{L}_p$ is in conjunctive normal form (CNF), if φ is of the form $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \varphi_{ij}$, where $n, m_1, \dots, m_n \in \mathbb{N}^+$ and each φ_{ij} is a propositional variable or a negation of a propositional variable.

We say that a formula φ is a *CNF-formula* if φ is in conjunctive normal form. Let CNF_p denote the set of all formulas $\varphi \in \mathcal{L}_p$ in CNF.

Proposition 12. There is a function $h : \mathcal{L}_p \rightarrow \text{CNF}_p$ such that for all $\varphi \in \mathcal{L}_p$, $\vdash_{\text{PL}} \varphi \leftrightarrow h(\varphi)$.

Proof. Such a function can be found in many textbooks (see e.g., [19, p. 221, Theorem 4.7]). \square

Definition 11. We say that $\varphi \in \mathcal{L}^-$ is *clean*, if there are $\psi_1, \dots, \psi_n \in \mathcal{L}_H$, $\gamma_1, \dots, \gamma_m \in \mathcal{L}_S$ and $\alpha(p_1^h, \dots, p_n^h, p_1^s, \dots, p_m^s) \in \mathcal{L}_p$ such that

$$\varphi = \alpha(\psi_1, \dots, \psi_n, \gamma_1, \dots, \gamma_m).$$

Moreover, if α is in CNF, then φ is called a *clean CNF-formula*. Let \mathcal{L}_c and CNF_c denote the set of all clean formulas and the set of all clean CNF-formulas, respectively.

Now, we present a Hilbert style calculus LHS^- for LHS^- , and details are as follows:

Proof system LHS^- for LHS^-	
Axiom schemes:	
(A1)	$\alpha \rightarrow (\beta \rightarrow \alpha)$
(A2)	$(\alpha \rightarrow (\beta \rightarrow \theta)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \theta))$.
(A3)	$(\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)$.
(K)	$[\star](\alpha \rightarrow \beta) \rightarrow ([\star]\alpha \rightarrow [\star]\beta)$, for $\star \in \{H, S\}$.
(R _H)	$[H](\psi \vee \gamma) \leftrightarrow ([H]\psi \vee \gamma)$, where $\psi \in \mathcal{L}_H$ and $\gamma \in \mathcal{L}_S$.
(R _S)	$[S](\psi \vee \gamma) \leftrightarrow (\psi \vee [S]\gamma)$, where $\psi \in \mathcal{L}_H$ and $\gamma \in \mathcal{L}_S$.
Inference rules:	
(MP)	From α and $\alpha \rightarrow \beta$, infer β .
(Nec _[\star])	From α , infer $[\star]\alpha$, for $[\star] \in \{H, S\}$.

It is a direct extension of the calculi PL , K_H and K_S . Therefore, we have the following:

Proposition 13. For all $\varphi \in \mathcal{L}_p$, if $\vdash_{\text{PL}} \varphi$, then $\vdash_{\text{LHS}^-} \varphi$.

Proposition 14. For any $\varphi \in \mathcal{L}_H$ and $\star \in \{H, S\}$, if $\vdash_{\text{K}_\star} \varphi$, then $\vdash_{\text{LHS}^-} \varphi$.

Now we proceed to show the following:

Corollary 3. There is a function $f : \mathcal{L}_c \rightarrow \text{CNF}_c$ s.t. for all $\varphi \in \mathcal{L}_c$, it holds that $\vdash_{\text{LHS}^-} \varphi \leftrightarrow f(\varphi)$. Also, the resulting formula $f(\varphi)$ is of the form $\bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$, where $\bigwedge_{i=1}^n \psi_i \in \mathcal{L}_H$ and $\bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_S$.

Proof. Let φ be a clean formula. Then, there are formulas $\alpha_1, \dots, \alpha_k \in \mathcal{L}_H \cup \mathcal{L}_S$ and $\beta(p_1, \dots, p_k) \in \mathcal{L}_p$ such that $\varphi = \beta(\alpha_1, \dots, \alpha_k)$. It follows from Proposition 12 that $\vdash_{\mathbf{PL}} \beta \leftrightarrow h(\beta)$. Note that $h(\beta)$ is in CNF, and so $h(\beta)$ is of the form $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \beta_{ij}$ with $n, m_1, \dots, m_n \in \mathbb{N}^+$. For each $1 \leq i \leq n$, we define

$$\begin{aligned} \psi_i &:= (p^h \wedge \neg p^h) \vee \bigvee \{\beta_{ij} \in \mathcal{L}_H : 1 \leq j \leq m_i\} \text{ and} \\ \gamma_i &:= (p^s \wedge \neg p^s) \vee \bigvee \{\beta_{ij} \in \mathcal{L}_S : 1 \leq j \leq m_i\}, \end{aligned}$$

where $p^h \in \mathbf{H}$ and $p^s \in \mathbf{S}$ are new propositional variables. Then, it holds that $\vdash_{\mathbf{PL}} \bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \beta_{ij} \leftrightarrow \bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$. By Proposition 13, $\vdash_{\mathbf{LHS}^-} \beta(p_1, \dots, p_k) \leftrightarrow \bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$. Then, we have the following:

$$\vdash_{\mathbf{LHS}^-} \varphi \leftrightarrow \bigwedge_{i=1}^n (\psi_i(\alpha_1, \dots, \alpha_k, p^h) \vee \gamma_i(\alpha_1, \dots, \alpha_k, p^s)).$$

Now, we can define a desired function $f : \mathcal{L}_c \rightarrow \text{CNF}_c$ as follows:

$$f(\varphi) = \bigwedge_{i=1}^n (\psi_i(\alpha_1, \dots, \alpha_k, p^h) \vee \gamma_i(\alpha_1, \dots, \alpha_k, p^s)),$$

which completes the proof. \square

Lemma 1. *Let $\mathfrak{M} = (W, R, V)$ be a model and $s, t \in W$. Then, for all formulas $\varphi \in \mathcal{L}_H$ and $\psi \in \mathcal{L}_S$, the following equivalences hold:*

- (1) $\mathfrak{M}, s, t \models \varphi$ if, and only if, for all $t' \in W$, $\mathfrak{M}, s, t' \models \varphi$.
- (2) $\mathfrak{M}, s, t \models \psi$ if, and only if, for all $s' \in W$, $\mathfrak{M}, s', t \models \psi$.
- (3) $\mathfrak{M}, s, t \models [\mathbf{H}](\varphi \vee \psi)$ if, and only if, $\mathfrak{M}, s, t \models [\mathbf{H}]\varphi \vee \psi$.
- (4) $\mathfrak{M}, s, t \models [\mathbf{S}](\varphi \vee \psi)$ if, and only if, $\mathfrak{M}, s, t \models \varphi \vee [\mathbf{S}]\psi$.

Proof. The proofs for (1) and (2) are by induction on the complexity of φ and ψ , respectively. In what follows, we merely consider for (3), and (4) can be proved in a similar way.

For the direction from left to right, we assume that $\mathfrak{M}, s, t \not\models [\mathbf{H}]\varphi \vee \psi$. Then, $\mathfrak{M}, s, t \models \neg\psi$ and there is some state $s' \in R(s)$ such that $\mathfrak{M}, s', t \not\models \varphi$. Note that $\neg\psi \in \mathcal{L}_S$. Now, using the item (2), we can obtain $\mathfrak{M}, s', t \models \neg\psi$. Thus, it holds that $\mathfrak{M}, s', t \not\models \varphi \vee \psi$, and so $\mathfrak{M}, s, t \not\models [\mathbf{H}](\varphi \vee \psi)$.

For the converse direction, we assume that $\mathfrak{M}, s, t \not\models [\mathbf{H}](\varphi \vee \psi)$. Then, there is some state $s' \in R(s)$ s.t. $\mathfrak{M}, s', t \not\models \varphi \vee \psi$, which entails $\mathfrak{M}, s, t \not\models [\mathbf{H}]\varphi$ and $\mathfrak{M}, s', t \not\models \psi$. Using item (2), we can infer $\mathfrak{M}, s, t \not\models \psi$ from $\mathfrak{M}, s', t \not\models \psi$. Hence, $\mathfrak{M}, s, t \not\models [\mathbf{H}]\varphi \vee \psi$. The proof is completed. \square

From the items (3) and (4) of Lemma 1, it is a matter of direct checking that:

Lemma 2. *For all $\varphi \in \mathcal{L}_H, \psi \in \mathcal{L}_S$, both the formulas $[\mathbf{H}](\varphi \vee \psi) \leftrightarrow ([\mathbf{H}]\varphi \vee \psi)$ and $[\mathbf{S}](\varphi \vee \psi) \leftrightarrow (\varphi \vee [\mathbf{S}]\psi)$ are valid.*

With the lemma, it is easy to obtain the soundness of the proof system \mathbf{LHS}^- :

Theorem 2. For any formula $\varphi \in \mathcal{L}^-$, $\vdash_{\mathbf{LHS}^-} \varphi$ implies $\models \varphi$.

Next, we consider for the completeness of the calculus \mathbf{LHS}^- .

Definition 12. For any $\varphi \in \mathcal{L}^-$, we define its clean CNF companion φ_c in the following:

$$\begin{aligned}
 (p^h)_c &:= p^h \vee (p_0^s \wedge \neg p_0^s), \text{ where } p_0^s \text{ is a new propositional variable from } \mathbf{S}. \\
 (p^s)_c &:= (p_0^h \wedge \neg p_0^h) \vee p^s, \text{ where } p_0^h \text{ is a new propositional variable from } \mathbf{H}. \\
 (\neg\varphi)_c &:= f(\neg\varphi_c) \\
 (\varphi \wedge \psi)_c &:= \varphi_c \wedge \psi_c \\
 ([\mathbf{H}]\varphi)_c &:= \bigwedge_{i=1}^n ([\mathbf{H}]\psi_i \vee \gamma_i), \text{ where } \bigwedge_{i=1}^n \psi_i \in \mathcal{L}_{\mathbf{H}}, \bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_{\mathbf{S}} \text{ and } \varphi_c = \bigwedge_{i=1}^n (\psi_i \vee \gamma_i). \\
 ([\mathbf{S}]\varphi)_c &:= \bigwedge_{i=1}^n (\psi_i \vee [\mathbf{S}]\gamma_i), \text{ where } \bigwedge_{i=1}^n \psi_i \in \mathcal{L}_{\mathbf{H}}, \bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_{\mathbf{S}} \text{ and } \varphi_c = \bigwedge_{i=1}^n (\psi_i \vee \gamma_i).
 \end{aligned}$$

Example 2. Let us consider an example $[\mathbf{S}]p^h$. It is easy to see that $([\mathbf{S}]p^h)_c = p^h \vee [\mathbf{S}](p_0^s \wedge \neg p_0^s)$. Note that if we define $(p^h)_c$ to be p^h , then we cannot ensure that a formula and its clean companion are equivalent: for instance, $[\mathbf{S}]p^h \leftrightarrow p^h$ is not a validity. Similarly for the clause of $p^s \in \mathbf{S}$.

Theorem 3. Let $\varphi \in \mathcal{L}^-$. Then, its clean CNF companion φ_c is of the form $\bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$ with $\bigwedge_{i=1}^n \psi_i \in \mathcal{L}_{\mathbf{H}}$ and $\bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_{\mathbf{S}}$. Moreover, it holds that $\vdash_{\mathbf{LHS}^-} \varphi \leftrightarrow \varphi_c$.

Proof. The first part of the theorem is easy, since φ_c is a clean CNF formula. In what follow, by induction on $\varphi \in \mathcal{L}^-$, we will show that $\vdash_{\mathbf{LHS}^-} \varphi \leftrightarrow \varphi_c$. The cases for propositional atoms and \wedge are straightforward, and we consider others.

(1). First, we consider the case that φ is $\neg\psi$. By the inductive hypothesis, $\vdash_{\mathbf{LHS}^-} \psi \leftrightarrow \psi_c$. So, $\vdash_{\mathbf{LHS}^-} \neg\psi \leftrightarrow \neg\psi_c$. Clearly, $\neg\psi_c \in \mathcal{L}_c$. From Corollary 3 we know that $\vdash_{\mathbf{LHS}^-} f(\neg\psi_c) \leftrightarrow \neg\psi_c$. Also, with the clause for \neg in Definition 12, we have $(\varphi)_c = f(\neg\psi_c)$. Immediately, $\vdash_{\mathbf{LHS}^-} \varphi_c \leftrightarrow \varphi$.

(2). Next, we move to the case that φ is $[\mathbf{H}]\psi$. Assume that $\psi_c = \bigwedge_{i=1}^n (\psi'_i \vee \gamma'_i)$, where $\bigwedge_{i=1}^n \psi'_i \in \mathcal{L}_{\mathbf{H}}$ and $\bigwedge_{i=1}^n \gamma'_i \in \mathcal{L}_{\mathbf{S}}$. Then, $\vdash_{\mathbf{LHS}^-} [\mathbf{H}]\psi_c \leftrightarrow \bigwedge_{i=1}^n [\mathbf{H}](\psi'_i \vee \gamma'_i)$. For simplicity, we write $\gamma(p^h, p^s)$ for $[\mathbf{H}](p^h \vee p^s) \leftrightarrow ([\mathbf{H}]p^h \vee p^s)$, which is exactly the axiom $(\mathbf{R}_{\mathbf{H}})$. Note that for each $1 \leq i \leq n$, $\psi'_i \in \mathcal{L}_{\mathbf{H}}$ and $\gamma'_i \in \mathcal{L}_{\mathbf{S}}$. So, for each $1 \leq i \leq n$, we can obtain $\vdash_{\mathbf{LHS}^-} \gamma(\psi'_i, \gamma'_i)$, i.e., $\vdash_{\mathbf{LHS}^-} [\mathbf{H}](\psi'_i \vee \gamma'_i) \leftrightarrow ([\mathbf{H}]\psi'_i \vee \gamma'_i)$. Therefore, $\vdash_{\mathbf{LHS}^-} \bigwedge_{i=1}^n [\mathbf{H}](\psi'_i \vee \gamma'_i) \leftrightarrow \bigwedge_{i=1}^n ([\mathbf{H}]\psi'_i \vee \gamma'_i)$, which entails $\vdash_{\mathbf{LHS}^-} [\mathbf{H}]\psi_c \leftrightarrow \varphi_c$. By the inductive hypothesis, $\vdash_{\mathbf{LHS}^-} \psi \leftrightarrow \psi_c$ and so $\vdash_{\mathbf{LHS}^-} \varphi \leftrightarrow [\mathbf{H}]\psi_c$. Hence $\vdash_{\mathbf{LHS}^-} \varphi \leftrightarrow \varphi_c$.

(3). Finally, the proof for the case that φ is $[\mathbf{S}]\psi$ is similar to (2). The proof is completed. \square

Example 3. Applications of Theorem 3 can be diverse. As an example, we show how to use it to prove $\vdash_{\mathbf{LHS}^-} [\mathbf{H}][\mathbf{S}]\neg\varphi \leftrightarrow [\mathbf{S}][\mathbf{H}]\neg\varphi$. Let $(\neg\varphi)_c = \bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$

with $\bigwedge_{i=1}^n \psi_i \in \mathcal{L}_H$ and $\bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_S$. By Theorem 3, $\vdash_{\mathbf{LHS}^-} \neg\varphi \leftrightarrow (\neg\varphi)_c$. Then, $\vdash_{\mathbf{LHS}^-} [S][H]\neg\varphi \leftrightarrow [S][H](\neg\varphi)_c$ and $\vdash_{\mathbf{LHS}^-} [H][S]\neg\varphi \leftrightarrow [H][S](\neg\varphi)_c$. Also, we have both $\vdash_{\mathbf{LHS}^-} [S][H](\neg\varphi)_c \leftrightarrow \bigwedge_{i=1}^n ([H]\psi_i \vee [S]\gamma_i)$ and $\vdash_{\mathbf{LHS}^-} [H][S](\neg\varphi)_c \leftrightarrow \bigwedge_{i=1}^n ([H]\psi_i \vee [S]\gamma_i)$. Thus, we obtain $\vdash_{\mathbf{LHS}^-} [H][S]\neg\varphi \leftrightarrow [S][H]\neg\varphi$.

Now, with the help of Theorem 3, we can show that the proof system \mathbf{LHS}^- is complete with respect to the class $\text{Mod}^{<\omega}$ of finite models:

Theorem 4. *For each $\varphi \in \mathcal{L}^-$, if $\text{Mod}^{<\omega} \models \varphi$ then $\vdash_{\mathbf{LHS}^-} \varphi$.*

Proof. Let $\varphi \in \mathcal{L}^-$ and $\not\vdash_{\mathbf{LHS}^-} \varphi$. By Theorem 3, $\not\vdash_{\mathbf{LHS}^-} \varphi_c$ and φ_c is of the form $\bigwedge_{i=1}^n (\psi_i \vee \gamma_i)$ where $\bigwedge_{i=1}^n \psi_i \in \mathcal{L}_H$ and $\bigwedge_{i=1}^n \gamma_i \in \mathcal{L}_S$. Since $\not\vdash_{\mathbf{LHS}^-} \varphi_c$, there is some i such that $\not\vdash_{\mathbf{LHS}^-} \psi_i \vee \gamma_i$. By Proposition 14, $\not\vdash_{\mathbf{K}_H} \psi_i$ and $\not\vdash_{\mathbf{K}_S} \gamma_i$. By the completeness of \mathbf{K}_H and \mathbf{K}_S , both $\neg\psi_i$ and $\neg\gamma_i$ are satisfiable. Since \mathbf{K}_H and \mathbf{K}_S have the finite model property [12], there are finite pointed \mathbf{K}_H -model (\mathfrak{M}, s) and finite pointed \mathbf{K}_S -model (\mathfrak{N}, t) such that $\mathfrak{M}, s \not\models \psi_i$ and $\mathfrak{N}, t \not\models \gamma_i$. Then, by Proposition 10, $\mathfrak{M} \uplus \mathfrak{N}, s, t \models \neg\psi_i \wedge \neg\gamma_i$, which entails $\mathfrak{M} \uplus \mathfrak{N}, s, t \models \neg\varphi_c$. By Theorems 2 and 3, $\text{Mod}^{<\omega} \not\models \varphi$. \square

Theorem 5. *\mathbf{LHS}^- enjoys the finite model property, and it is decidable.*

Proof. Assume that $\varphi \in \mathcal{L}^-$ is satisfiable. By Theorem 2, we have $\not\vdash_{\mathbf{LHS}^-} \neg\varphi$. Then, it follows from Theorem 4 that there is some finite model \mathfrak{M}' satisfying φ . So, the first claim holds. Now, since \mathbf{LHS}^- can be finitely axiomatized and has the finite model property, the logic is decidable. \square

Finally, it is worth noting that our proofs for the completeness and the decidability depend heavily on the fact that we can separate the language \mathcal{L}^- into two ‘isolated’ parts, which would not be the case if we have the constant \mathcal{I} in our language. This means we cannot transfer the results for \mathbf{LHS}^- to \mathbf{LHS} . Some efforts for the axiomatization of \mathbf{LHS} will be made in Section 6, and in the next section, we will first consider for the computational behavior of \mathbf{LHS} .

5 Computational behavior of \mathbf{LHS}

Essentially, \mathcal{I} is a propositional constant dealing with equality in a modal framework. This universally accepted relation of indiscernibility is simple in nature. However, as we mentioned in Section 1, there are various elegant examples of logics that suggest that taking this relation into account may change previously decidable logics (without equality) into undecidable ones. In this section, we are going to contribute one more instance to this class: in what follows, we first show that \mathbf{LHS} does not have the tree model property or the finite model property, and then prove that the satisfiability problem for \mathbf{LHS} is undecidable. Then, by considering the relations between \mathbf{LHS} and other relevant logics, we identify the complexity of the model checking problem for our logic, and it turns out that the model checking problem for \mathbf{LHS} is P-complete, which would also give us an

upper bound for determining the winner in a given game of hide and seek (with a finite graph).¹⁴

5.1 Undecidability of LHS

Usually, the tree model property and the finite model property are positive signals for the computational behavior of a logic (cf. e.g., [12]). As a warm-up, we show that LHS lacks the properties.

Proposition 15. *LHS does not have the tree model property.*

Proof. Consider the formula $\mathcal{I} \wedge \langle \mathbf{H} \rangle \mathcal{I}$ that is satisfiable. Also, let $\mathfrak{M} = (W, R, V)$ and $u, v \in W$ such that $\mathfrak{M}, u, v \models \mathcal{I} \wedge \langle \mathbf{H} \rangle \mathcal{I}$. From \mathcal{I} it follows that $u = v$. Also, the conjunct $\langle \mathbf{H} \rangle \mathcal{I}$ indicates that $u \in R(u)$. Consequently, \mathfrak{M} cannot be a tree. The proof is completed. \square

Moreover, by constructing a ‘spy-point’,¹⁵ we can also prove the following:

Theorem 6. *LHS lacks the finite model property.*

Proof. Let φ_∞ be the conjunction of the following formulas:

$$(F1) \mathcal{I} \wedge [\mathbf{H}] \neg \mathcal{I} \quad (F2) \langle \mathbf{H} \rangle [\mathbf{H}] \perp \quad (F3) [\mathbf{H}] \langle \mathbf{S} \rangle (\neg \mathcal{I} \wedge \langle \mathbf{S} \rangle \top \wedge [\mathbf{S}] \mathcal{I})$$

Let us briefly comment on the intuition underlying these formulas. First, (F1) shows that the two states of the current pointed LHS-model are the same and the point is irreflexive. Also, formula (F2) states that the point can reach a state that is a dead end having no successors. Additionally, (F3), motivated by [31], indicates that the point has more than one successor and for all its successors i , there is also another successor j such that j has i as its only successor. The formula φ_∞ is satisfiable: for instance, it is true at (s, s) in the model \mathfrak{M}_∞ depicted in Figure 3.

Next, let $\mathfrak{M} = (W, R, V)$ be a model and $u \in W$ such that $\mathfrak{M}, u, u \models \varphi_\infty$. We are going to show that W is infinite. To do so, we claim that the model contains the following sequence of states of \mathfrak{M} :

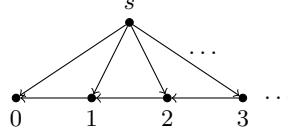
$$w_0, w_1, w_2, w_3, w_4, \dots$$

such that for all $i \in \mathbb{N}$, the following conditions hold:

$$\text{P1. } \mathfrak{M}, w_i, w_{i+1} \models \neg \mathcal{I} \wedge \langle \mathbf{S} \rangle \top \wedge [\mathbf{S}] \mathcal{I}$$

¹⁴ For a set A , we use $|A|$ for its *cardinality*. Given a finite graph (W, R) , Seeker can win iff she can win in $|W|^2$ rounds. So, determining the winner in the game equates to checking whether the formula $\bigvee_{n \leq |W|^2} ([\mathbf{H}] \langle \mathbf{S} \rangle)^n \mathcal{I}$ is true at (s, t) in the model (W, R, V) , where s and t are the positions of Hider and Seeker respectively, and V is an arbitrary valuation function.

¹⁵ A key feature of a spy point is that all states that are reachable from it in n -steps can also be reached in one step, which is developed by [11] to show the undecidability of some hybrid logics.

Fig. 3. The model \mathfrak{M}_∞ .

P2. $(u, w_i) \in R$

P3. $R(w_0) = \emptyset$, and for $1 \leq i$, $R(w_i) = \{w_{i-1}\}$

By making an induction on $i \in \mathbb{N}$, we show that there is always such a sequence of those w_i .

First, let us consider the basic case that $i = 0$. As $\mathfrak{M}, u, u \models (F2)$, we know that there is $w_0 \in W$ such that Ruw_0 and $\mathfrak{M}, w_0, u \models [H]\perp$. Therefore, $R(w_0) = \emptyset$. Moreover, by formula (F3), it holds $\mathfrak{M}, w_0, u \models \langle S \rangle (\neg \mathcal{I} \wedge \langle S \rangle \top \wedge [S]\mathcal{I})$. Therefore, there exists $w_1 \in W$ such that $Ruw_1, w_0 \neq w_1$ and $R(w_1) = \{w_0\}$. Now, it is not hard to see that the clauses P1-P3 hold for both w_0 and w_1 .

Now, suppose that we have already had all those states $w_{i \leq n}$, and we proceed to show that there exists w_{n+1} satisfying the conditions P1-P3. By the inductive hypothesis, we have Ruw_n . Now, from the formula (F3), it follows that $\mathfrak{M}, w_n, u \models \langle S \rangle (\neg \mathcal{I} \wedge \langle S \rangle \top \wedge [S]\mathcal{I})$. So, there is a state $w_{n+1} \in W$ such that Ruw_{n+1} and $\mathfrak{M}, w_n, w_{n+1} \models \neg \mathcal{I} \wedge \langle S \rangle \top \wedge [S]\mathcal{I}$. This indicates that w_{n+1} satisfies the requirements P1 and P2. Also, as $\neg \mathcal{I}$, $w_n \neq w_{n+1}$. Furthermore, from $\mathfrak{M}, w_n, w_{n+1} \models \langle S \rangle \top \wedge [S]\mathcal{I}$, we know that $R(w_{n+1}) = \{w_n\}$, which indicates that the node satisfies P3 as well.

Moreover, it follows from P1 and P3 that whenever $i \neq j$, $w_i \neq w_j$. To be more specific, we have $\mathfrak{M}, w_i, w_i \models \langle H \rangle^i \top \wedge [H]^{i+1} \perp$ for each i . Thus, there are infinitely many states. So, \mathfrak{M} is infinite. \square

Theorem 7. *The satisfiability problem for LHS is undecidable.*

Proof. We show this by reduction of the $\mathbb{N} \times \mathbb{N}$ tiling problem [41] to the satisfiability problem for LHS. Let $\mathbb{T} = \{T_1, \dots, T_n\}$ be some fixed set of tile types. For each $T_i \in \mathbb{T}$, we use $\text{up}(T_i)$, $\text{down}(T_i)$, $\text{left}(T_i)$ and $\text{right}(T_i)$ to represent the colors of its up, down, left and right edges, respectively. We say that \mathbb{T} tiles $\mathbb{N} \times \mathbb{N}$ if there is a function $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{T}$ such that for all $n, m \in \mathbb{N}$,

$$\text{right}(g(n, m)) = \text{left}(g(n+1, m)) \text{ and } \text{up}(g(n, m)) = \text{down}(g(n, m+1)).$$

Functions satisfying the conditions above are called *tiling functions*. In what follows, to show that LHS is undecidable, we present a formula $\varphi_{\mathbb{T}}$ such that $\varphi_{\mathbb{T}}$ is satisfiable if and only if \mathbb{T} tiles $\mathbb{N} \times \mathbb{N}$.

Let $\text{Label} := \{u, r\} \cup \{t_i : 1 \leq i \leq n\}$ be a finite set of labels. Based on it, let $\text{VH} := \{p^h : p \in \text{Label}\}$ and $\text{VS} := \{p^s : p \in \text{Label}\}$ be two sets of new variables. Also, we denote $\bigvee_{1 \leq i \leq n} t_i^h$ by t^h and $\bigvee_{1 \leq i \leq n} t_i^s$ by t^s . We write $\langle H \rangle_u \varphi$ for formula $t^h \wedge \langle H \rangle (u^h \wedge \langle H \rangle (t^h \wedge \varphi))$ and $\langle H \rangle_r \varphi$ for $t^h \wedge \langle H \rangle (r^h \wedge \langle H \rangle (t^h \wedge \varphi))$. Operators

$\langle S \rangle_u$ and $\langle S \rangle_r$ are defined similarly. Their dual operators are defined as usual, e.g., $[\mathbf{H}]_u \varphi := \neg \langle \mathbf{H} \rangle_u \neg \varphi$.

The formula $\varphi_{\mathbb{T}}$ is the conjunction of those in the groups below. To facilitate discussion, let $\mathfrak{M} = (W, R, V)$ be a model and $w, v \in W$ s.t. $\mathfrak{M}, w, v \models \varphi_{\mathbb{T}}$.

Group 1 (Basic requirements):

- (SP) $\mathcal{I} \wedge [\mathbf{H}][\mathbf{H}]\langle \mathbf{S} \rangle \mathcal{I} \wedge \langle \mathbf{H} \rangle t^h$
- (VL1) $[\mathbf{H}][\mathbf{S}](\mathcal{I} \rightarrow \bigwedge_{p \in \text{Label}} (p^h \leftrightarrow p^s))$
- (VL2) $[\mathbf{H}] \bigwedge_{p \in \text{Label}} (p^h \leftrightarrow \bigwedge_{p \neq q \in \text{Label}} \neg q^h)$

Let us explain the meanings of the formulas in Group 1. Intuitively, we can treat t, u, r as labels. The formula (SP) says that $w = v$, $R(R(w)) \subseteq R(w)$ and there is some $v \in R(w)$ which is labelled by some t_i . (VL1) indicates that for any $s \in R(w)$, its label for Hider and label for Seeker are always the same. Moreover, (VL2) shows that every point $s \in R(w)$ has exactly one label.

Group 2 (Grid requirements):

- (TU1) $[\mathbf{H}][\mathbf{S}](t^h \wedge \mathcal{I} \rightarrow \langle \mathbf{H} \rangle (u^h \wedge [\mathbf{S}](u^s \rightarrow \mathcal{I})))$
- (TU2) $[\mathbf{H}][\mathbf{S}](u^h \wedge \mathcal{I} \rightarrow \langle \mathbf{H} \rangle (t^h \wedge [\mathbf{S}](t^s \rightarrow \mathcal{I})))$
- (TR1) $[\mathbf{H}][\mathbf{S}](t^h \wedge \mathcal{I} \rightarrow \langle \mathbf{H} \rangle (r^h \wedge [\mathbf{S}](r^s \rightarrow \mathcal{I})))$
- (TR2) $[\mathbf{H}][\mathbf{S}](r^h \wedge \mathcal{I} \rightarrow \langle \mathbf{H} \rangle (t^h \wedge [\mathbf{S}](t^s \rightarrow \mathcal{I})))$
- (URT) $[\mathbf{H}][\mathbf{S}](t^h \wedge \mathcal{I} \rightarrow [\mathbf{H}]_u [\mathbf{S}]_r \langle \mathbf{H} \rangle_r \langle \mathbf{S} \rangle_u \mathcal{I})$

Without loss of generality, we assume that \mathfrak{M} is a model generated by $w \in W$ (Proposition 1). Let

$$R_u := \{(s, t) \in R : \mathfrak{M}, s, t \models t^h \wedge t^s \text{ and } V(u^h) \cap R(s) \cap R^{-1}(t) \neq \emptyset\}.$$

It follows from (TU1) and (TU2) that for all $s \in R(w)$, $|R_u(s)| = 1$. Similarly, we can define R_r , and by (TR1) and (TR2), for all $s \in R(w)$, $|R_r(s)| = 1$. From (URT), we can infer that for all $v \in R(w)$, $R_r(R_u(v)) = R_u(R_r(v))$.

Group 3 (Tiling the model):

- (T1) $[\mathbf{H}](t^h \rightarrow \bigwedge_{i=1}^n (t_i^h \rightarrow \langle \mathbf{H} \rangle_u \bigvee_{1 \leq j \leq n \text{ \& up}(T_i)=\text{down}(T_j)} t_j^h))$
- (T2) $[\mathbf{H}](t^h \rightarrow \bigwedge_{i=1}^n (t_i^h \rightarrow \langle \mathbf{H} \rangle_r \bigvee_{1 \leq j \leq n \text{ \& right}(T_i)=\text{left}(T_j)} t_j^h))$

The formulas in Group 3 are standard, which tell us that \mathbb{T} ‘tiles’ $R(w) \cap V(t^h)$.

Lemma 3. *If \mathbb{T} tiles $\mathbb{N} \times \mathbb{N}$, then $\varphi_{\mathbb{T}}$ is satisfiable.*

Proof. Let $h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{T}$ be a tiling function. Define $\mathfrak{M}_h = (W, R, V)$ as follows:

- $W = W_0 \cup \{s\}$, where $W_0 = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \times m \text{ is even}\}$
- $R = R_r \cup R_u \cup (\{s\} \times W_0)$, where
 - $R_r = \{((k, 2l), (k+1, 2l)) : k, l \in \mathbb{N}\}$
 - $R_u = \{((2k, l), (2k, l+1)) : k, l \in \mathbb{N}\}$
- V is a valuation such that for all $1 \leq i \leq n$:
 - $V(r^h) = V(r^s) = \{(2k+1, 2l) \in W : k, l \in \mathbb{N}\}$,

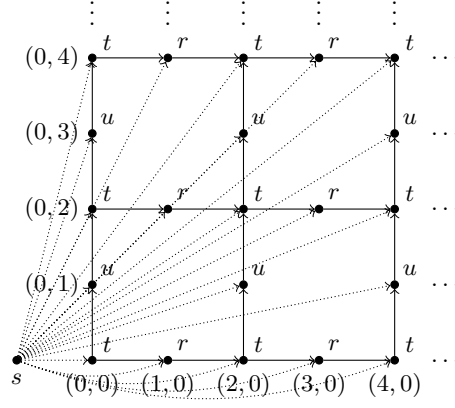


Fig. 4. \mathfrak{M}_h : Both dotted arrows and solid arrows represent the relation R .

- $V(u^h) = V(u^s) = \{(2k, 2l + 1) \in W : k, l \in \mathbb{N}\}$,
- $V(t_i^h) = V(t_i^s) = \{(2k, 2l) \in W : k, l \in \mathbb{N}, h(k, l) = T_i\}$,
- $V(p^h) = V(p^s) = \emptyset$ for all other $p^h, p^s \in \mathbf{H} \cup \mathbf{S}$.

The model \mathfrak{M}_h is shown in Figure 4. It is easy to verify that $\mathfrak{M}_h, s, s \models \varphi_{\mathbb{T}}$. \square

Lemma 4. *If $\varphi_{\mathbb{T}}$ is satisfiable, then \mathbb{T} tiles $\mathbb{N} \times \mathbb{N}$.*

Proof. Suppose that $\mathfrak{M} = (W, R, V)$ is a model generated by $s \in W$ and $\mathfrak{M}, s, s \models \varphi_{\mathbb{T}}$ (Proposition 1). It suffices to define a tiling function $g : \mathbb{N} \times \mathbb{N} \rightarrow T$. By (SP1), $V(t^h) \neq \emptyset$. Also, it follows from (TU1) and (TU2) that for each $w \in V(t^h)$, there is exactly one state $v \in V(t^h)$ such that Rwx and Rxv for some $x \in V(u^h)$, and we denote the state v by $\text{up}(w)$. Then $\text{up} : V(t^h) \rightarrow V(t^h)$ is a function. Similarly, due to (TR1) and (TR2), we can define a function $\text{right} : V(t^h) \rightarrow V(t^h)$. Let $w_0 \in V(t^h)$. We inductively define a function $g : \mathbb{N} \times \mathbb{N} \rightarrow V(t^h)$ as follows:

$$g(0, 0) = w_0, \quad g(n, m + 1) = \text{up}(g(n, m)), \quad g(n + 1, m) = \text{right}(g(n, m)).$$

By (URT), for each $w \in V(t^h)$, we have $\text{up}(\text{right}(w)) = \text{right}(\text{up}(w))$. Then, for all $(n, m) \in \mathbb{N} \times \mathbb{N}$,

$$\text{up}(g(n + 1, m)) = \text{up}(\text{right}(g(n, m))) = \text{right}(\text{up}(g(n, m))) = \text{right}(g(n, m + 1)).$$

Hence g is well-defined. Let $h : V(t^h) \rightarrow T$ be the function s.t. for each $1 \leq i \leq n$, $h(w) = T_i$ iff $w \in V(t_i^h)$. Finally, by the formulas in Group 3, $h \circ g : \mathbb{N} \times \mathbb{N} \rightarrow T$ is a tiling function. \square

Now the proof for the undecidability is completed. \square

Algorithm 1: LHS model checking, where a syntactic-growing sequence is a sequence of formulas $(\varphi_1, \dots, \varphi_n)$ such that the size of φ_i is no larger than that of φ_j whenever $i < j$.

Input: φ : a formula, $\mathfrak{M} = (W, R, V)$: a model

Output: The truth set of φ in \mathfrak{M}

```

1 function Truth( $\varphi, \mathfrak{M}$ )
2   Record all sub-formulas of  $\varphi$  with a syntactic-growing sequence  $S(\varphi)$ 
3   Record all truth sets of formulas in  $S(\varphi)$ , one by one, in a table truth as
   follows:
4   forall  $\psi \in S(\varphi)$  do
5      $truth(\psi) \leftarrow W$ 
6     if  $\psi \in \mathbf{H}$  then
7        $truth(\psi) \leftarrow V(\psi) \times W$ 
8     if  $\psi \in \mathbf{S}$  then
9        $truth(\psi) \leftarrow W \times V(\psi)$ 
10    if  $\psi = \mathcal{I}$  then
11       $truth(\psi) \leftarrow \{(s, s) : s \in W\}$ 
12    if  $\psi = \neg\chi$  then
13       $truth(\psi) \leftarrow (W \times W) \setminus truth(\chi)$ 
14    if  $\psi = \psi_1 \wedge \psi_2$  then
15       $truth(\psi) \leftarrow truth(\psi_1) \cap truth(\psi_2)$ 
16    if  $\psi = [\mathbf{H}]\chi$  then
17      forall  $(s', t) \notin truth(\chi)$  and  $s \in W$  do
18        if  $Rss'$  then
19           $truth(\psi) \leftarrow truth(\psi) \setminus \{(s, t)\}$ 
20    if  $\psi = [\mathbf{S}]\chi$  then
21      forall  $(s, t') \notin truth(\chi)$  and  $t \in W$  do
22        if  $Rtt'$  then
23           $truth(\psi) \leftarrow truth(\psi) \setminus \{(s, t)\}$ 
24  return  $truth(\varphi)$ 

```

5.2 Complexity of the model checking problem for LHS

To identify the complexity of the model checking problem for LHS, we are going to establish the bounds for the same. We start by showing a desired lower bound, and define a translation $t : \mathcal{L}_\square \rightarrow \mathcal{L}$ as follows: $t(p_i) := p_i^h$, for each propositional letter p_i ; the function preserves Boolean connectives \neg and \wedge ; and $t(\Box\varphi) := [H]t(\varphi)$. Also, a model $\mathfrak{M} = (W, R, V)$ for K_\square gives rise to a model $\mathfrak{M}^+ = (W, R, V^+)$ for LHS, where $V^+(p_i^h) := V(p_i)$ for each propositional letter p_i of \mathcal{L}_\square and $V^+(p^s) = \emptyset$. By induction on \mathcal{L}_\square , one can easily show the following:

Proposition 16. *Let $\mathfrak{M} = (W, R, V)$ be a model for the basic modal logic. Then for any $\varphi \in \mathcal{L}_\square$ and any $s, t \in W$, $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}^+, s, t \models t(\varphi)$.*

Our next step is to consider translation from \mathcal{L} to \mathcal{L}^1 . Since LHS is undecidable, LHS is not contained in the two-variable fragment of FOL. However, we have already a translation from LHS to the fragment of first-order logic with 3 variables, which is useful in showing the following:

Theorem 8. *Model checking for LHS is P-complete.*

Proof. A lower bound is provided by Proposition 16: the model checking for LHS is P-hard, since model checking for K_\square is P-complete [5]. On the other hand, as proved by [47], the model checking for every finite variable fragments of FOL, with a fixed number of variables, is in P. So, Corollary 1 establishes an upper bound for us. Thus, the model checking problem for LHS is P-complete. \square

Algorithm 1 presents a method to obtain the truth set of a formula φ in a given model \mathfrak{M} in $O(|\varphi| \times |W|^3)$ time, and it is involved with the length $|\varphi|$ of formulas φ , which is defined as follows: $|p^h| = |p^s| = |Z| = 1$, $|\neg\varphi| = |[H]\varphi| = |[S]\varphi| = |\varphi| + 1$ and $|\varphi \wedge \psi| = |\varphi| + |\psi| + 1$.

6 Zoom out: LHS as a product logic

We now move on to put LHS in the landscape of product logics (see, e.g., [37, 22, 23, 32]). More concretely, we will first identify the counterpart of LHS in product logics based on a sub-class of the so called ‘extended product models’. The formulation is natural, but restrictions are imposed on the sub-class that cannot be captured by the logic. Some restrictions are relaxed to build the logic on a larger class of models. We will relate LHS with product logics by showing representation results for our models in terms of certain product models. Although this is just a first step, these results may help to obtain other results for LHS, e.g., a complete Hilbert style proof system.

6.1 General setting: product models

First of all, to interpret \mathcal{L} properly, we build a class of models on product models. Given two frames $\mathcal{F}_1 = (W_1, R_1)$ and $\mathcal{F}_2 = (W_2, R_2)$, their *product frame* is $(W_1 \times W_2, R^l, R^r)$ with the following:

$$\begin{aligned} R^l(u, v)(s, t) &\text{ iff } R_1us \text{ and } v = t \\ R^r(u, v)(s, t) &\text{ iff } R_2vt \text{ and } u = s \end{aligned}$$

With this, we define the following enrichment:

Definition 13. An extended product model $\mathbb{M} = (S \times U, R^l, R^r, \mathbb{I}, \mathbb{V})$ for LHS is a tuple such that

- (1). $(S \times U, R^l, R^r)$ is a product frame,
- (2). $\mathbb{I} = \{(s, s) : s \in S \cap U\}$, and
- (3). $\mathbb{V} : \mathbf{H} \cup \mathbf{S} \rightarrow 2^{S \times U}$ s.t. $(s, t) \in \mathbb{V}(p^s)$ iff $S \times \{t\} \subseteq \mathbb{V}(p^s)$ and $(s, t) \in \mathbb{V}(p^h)$ iff $\{s\} \times U \subseteq \mathbb{V}(p^h)$.

We call tuples $(S \times U, R^l, R^r, \mathbb{I})$ without valuation functions extended product frames.

Truth conditions w.r.t. extended product models are straightforward:

$$\begin{aligned} \mathbb{M}, (s, t) \models p &\Leftrightarrow (s, t) \in \mathbb{V}(p), \quad \text{for each } p \in \mathbf{H} \cup \mathbf{S} \\ \mathbb{M}, (s, t) \models \mathcal{I} &\Leftrightarrow (s, t) \in \mathbb{I} \\ \mathbb{M}, (s, t) \models [\mathbf{H}]\varphi &\Leftrightarrow \mathbb{M}, (s', t') \models \varphi \text{ for all } (s', t') \in R^l((s, t)) \\ \mathbb{M}, (s, t) \models [\mathbf{S}]\varphi &\Leftrightarrow \mathbb{M}, (s', t') \models \varphi \text{ for all } (s', t') \in R^r((s, t)) \end{aligned}$$

In what follows, we are mainly interested in those extended product models with certain properties. Having LHS in mind, a natural and direct class of such models, denoted by \mathbb{G} , is as follows:

$$S = U \quad \& \quad R^l(s_1, t)(s_2, t) \text{ iff } R^r(t, s_1)(t, s_2).$$

Thus, an extended product model $(S \times U, R^l, R^r, \mathbb{I}, \mathbb{V})$ belongs to \mathbb{G} if, and only if, the part $(S \times U, R^l, R^r)$ is the product frame of some (S, R) and itself. The following show that there is a precise match between our original models and \mathbb{G} .

From standard models to \mathbb{G} . A standard model \mathfrak{M} can give rise to an extended product model, denoted by $\mathbb{M}^{\mathfrak{M}}$, in the following way:

Definition 14. Let $\mathfrak{M} = (W, R, V)$ be a standard model. The corresponding extended product model $\mathbb{M}^{\mathfrak{M}}$ is a tuple $(W \times W, R^l, R^r, \mathbb{I}, \mathbb{V})$ where

- $(W \times W, R^l, R^r)$ is the resulting product frame of (W, R) and itself,
- $\mathbb{I} = \{(w, w) : w \in W\}$,
- $(s, t) \in \mathbb{V}(p^h)$ iff $s \in V(p^h)$, and $(s, t) \in \mathbb{V}(p^s)$ iff $t \in V(p^s)$.

Obviously, $\{\mathbb{M}^{\mathfrak{M}} : \mathfrak{M} \text{ is a standard model}\} \subseteq \mathbb{G}$. By induction on formulas, we can show that:

Proposition 17. Let $\mathfrak{M} = (W, R, V)$ be a standard model and $s, t \in W$. Then, for all formulas $\varphi \in \mathcal{L}$, $\mathfrak{M}, s, t \models \varphi$ iff $\mathbb{M}^{\mathfrak{M}}, (s, t) \models \varphi$.

From \mathbb{G} to standard models. Also, every extended product model of \mathbb{G} is associated with a standard one in the following:

Definition 15. Let $\mathbb{M} = (W \times W, R^l, R^r, \mathbb{I}, \mathbb{V})$ be a \mathbb{G} -model. Then, we can obtain a standard model $\mathfrak{M}^{\mathbb{M}} = (W, R, V)$ such that:

- $R s_1 s_2$ iff $R^l(s_1, t)(s_2, t)$ and $R^r(t, s_1)(t, s_2)$ for all $t \in W$,
- $s \in V(p^h)$ iff $(s, t) \in \mathbb{V}(p^h)$ for all $t \in W$, and $s \in V(p^s)$ iff $(t, s) \in \mathbb{V}(p^s)$ for all $t \in W$.

It is simple to see that the resulting $\mathfrak{M}^{\mathbb{M}}$ is well-defined. With the above definition, we have:

Proposition 18. Let $\mathbb{M} = (W \times W, R^l, R^r, \mathbb{I}, \mathbb{V}) \in \mathbb{G}$. For all $\varphi \in \mathcal{L}$, it holds that: $\mathbb{M}, (s, t) \models \varphi$ iff $\mathfrak{M}^{\mathbb{M}}, s, t \models \varphi$.

Proof. We prove by induction on formulas. We merely show the case for $[\mathbf{H}]\varphi$.

Assume that $\mathbb{M}, (s, t) \not\models [\mathbf{H}]\varphi$. Then, there exists $(s', t) \in W \times W$ with $R^l(s, t)(s', t)$ and $\mathbb{M}, (s', t) \not\models \varphi$. Moreover, for all $t_1 \in W$, $R^l(s, t_1)(s', t_1)$ and $R^r(t_1, s)(t_1, s')$. With Definition 15, it holds $R s s'$. By the inductive hypothesis, $\mathfrak{M}^{\mathbb{M}}, s', t \not\models \varphi$. Therefore, $\mathfrak{M}^{\mathbb{M}}, s, t \not\models [\mathbf{H}]\varphi$.

For the other direction, suppose that $\mathfrak{M}^{\mathbb{M}}, s, t \not\models [\mathbf{H}]\varphi$. By the semantics (w.r.t. standard models), there exists $s' \in R(s)$ such that $\mathfrak{M}^{\mathbb{M}}, s', t \not\models \varphi$. By the inductive hypothesis, $\mathbb{M}, (s', t) \not\models \varphi$. Moreover, $R s s'$ implies that $R^l(s, t)(s', t)$ (recall Definition 15). So, $\mathbb{M}, (s, t) \not\models [\mathbf{H}]\varphi$. \square

From Propositions 17 and 18, it follows directly that

Theorem 9. The logic given by \mathbb{G} is exactly LHS.

In the sense above, the class of standard models is equivalent to \mathbb{G} . Thus, the product framework provides us a new angle to view our logic.

6.2 ‘Asymmetrizing’ domains: rectangle frames

However, it is important to notice that w.r.t. LHS, the class \mathbb{G} has many features that cannot be defined by the logic, e.g., the system cannot define the property that the domain is a product of a set with itself. Thus, we propose the following notion of ‘rectangle frames’:¹⁶

Definition 16. Let frames $\mathcal{F}_1 = (S, R_1)$ and $\mathcal{F}_2 = (U, R_2)$ be two frames and $(S \times U, R^r, R^l)$ be their product frame. A rectangle frame $\mathbb{F} = (S \times U, R^l, R^r, \mathbb{I})$ is an extended product frame satisfying:

$$\text{For all } w \in S \cap U, R_1(w) = R_2(w) \subseteq S \cap U.$$

We denote by \mathbb{R} the class of rectangle frames.

Notice that a point-generated frame of an \mathbb{R} -frame is in \mathbb{R} as well, and we in effect can identify \mathbb{R} with their point-generated frames. So, in what follows, we just work with the latter class.

¹⁶ To simplify the discussion, in this section we focus on *frames*.

Theorem 10. *The logic captured by \mathbb{G} -frames is the same as that of \mathbb{R} .*

Proof. It is easy to see that each \mathbb{G} -frame is an \mathbb{R} -frame. The crucial part is to show that *each \mathbb{R} -frame is a point-generated frame of some \mathbb{G} -frame*. To prove this, we show how to construct a \mathbb{G} -frame $\mathcal{G} = (W \times W, R^l, R^r, \mathbb{I})$ from a given \mathbb{R} -frame $\mathcal{R} = (S \times U, R_1^l, R_2^r, \mathbb{I}')$. Details are as follows:

- $W := S \cup U$
- For all $(s_1, t_1), (s_2, t_2) \in W \times W$,
 $R^l(s_1, t_1)(s_2, t_2)$ iff $t_1 = t_2$ and $(R_1s_1s_2$ or $R_2s_1s_2)$; and
 $R^r(s_1, t_1)(s_2, t_2)$ iff $s_1 = s_2$ and $(R_1t_1t_2$ or $R_2t_1t_2)$.
- $(s_1, t_1) \in \mathbb{I}$ iff $s_1 = t_1$.

One can check that the resulting frame is a \mathbb{G} -frame. Now we denote by (m, n) the root of \mathcal{R} , and let us proceed to show that \mathcal{R} is a point-generated frame of \mathcal{G} . To achieve our goal, we just need to show that a state $(s, t) \in \mathcal{G}$ that can be reached from (m, n) in one step is a state of \mathcal{R} as well: repeating the reasoning, we can show that all states of \mathcal{G} that are reached from (m, n) in i steps are also states of \mathcal{R} . We just show the case for $R^l(m, n)(s, t)$, and that for R^r is similar.

Immediately, $n = t$. Also, we have R_1ms or R_2ms . If R_2ms is the case, then from $(m, n) \in S \times U$ it follows $m \in S \cap U$. Thus, by the definition of \mathbb{R} , R_2ms implies that R_1ms . So, it suffices to consider R_1ms only. Now, by the definition of R_1^l , it holds $R_1^l(m, n)(s, n)$. This completes the proof. \square

Finally, notice that many earlier notions and results for LHS can be transferred into this new setting, and conversely, there are a large number of general results and techniques for product logics, and it remains to investigate which ones can transfer to our logic. We believe the connections established in this section may shed light on further study of LHS, e.g., its axiomatization, that is left as future work:

Open problem. Is LHS finitely axiomatizable?

A possible way to answer this might be to analyze the counterpart of LHS in product logic, for which the notion of rectangle frames and its possible further generalizations might be a starting point. Also, another direction might be to consider for the further enrichments of LHS, and as proved in [43], we can have a desired finite axiomatization after extending LHS with formulas of hybrid logic.

7 Related works

Graph games and modal logics. Motivated by a simple graph game of hide and seek, this work belongs to a broader program [7] that promotes a study of graph game design in tandem with matching new modal logics. As stated earlier, this chapter is an extension of [35, 36, 16]. So far, both LHS^- and LHS have been extended with ingredients from hybrid logic [43], and the resulting logics have many desired properties, including finite axiomatization. In recent years, several

interesting graph games have been studied. For instance, in *sabotage games* [6], a player moves along a link available to her on a graph to reach some fixed goal region, while her opponent removes an arbitrary link in each round to prevent her from reaching her goal. The games are captured by the *sabotage modal logic SML*, which is proposed in [8], and since then its logical properties have been studied in detail: we refer to [1, 42] for an in-depth exploration on the topic, to [10] for the axiomatization of SML and to [24] for complexity of the bisimulation problem for the logic.

Games in which links are removed *locally* according to certain conditions which are expressed explicitly in the language have been studied in [33] and [17]. Moreover, several variants of sabotage games are applied to the learning/teaching scenarios [25], and their computational behaviors are analyzed. Following this direction, a game setting allowing both link deletion and link addition is developed in [4] to capture some interesting features of the learning process. Instead of modifying links, in poison games [18], a player can poison a node to make it unavailable to the opponent. These are studied independently with diverse modal approaches in [13, 27]. Additionally, by updating valuation functions of models, a dynamic logic of local fact change is developed in [45], which captures a class of graph games in which properties of states might get affected by those of others.

Product logics with diagonal constant. Technically, our proposal is close to product logics [37, 22], especially to those with the diagonal constant δ [29, 31, 28]. What we mainly focus on is the framework $\mathsf{K} \times^\delta \mathsf{K}$.¹⁷ A crucial difference between our logic and $\mathsf{K} \times^\delta \mathsf{K}$ is that LHS have two kinds of propositional variables, one for each dimension. This feature has interesting consequences (cf. Section 4). In what follows, we will compare our logic with $\mathsf{K} \times^\delta \mathsf{K}$ as presented in [31, 28]: $\mathsf{K} \times^\delta \mathsf{K}$ lacks the finite model property and is undecidable, which seem very similar to our results at a first glance. However, our logic differs from this one on both conceptual and technical levels.

First, the techniques used to show the undecidability of LHS are very different. Similar to all other product logics, various relations representing transitions of states in different dimensions are considered in [31, 28]. Moreover, the product nature endows the relations with possible interactions: say, *commutativity* and *confluence*. With such interactions, product logics obtain grid-like structures automatically. However, a crucial step in our proof for the undecidability of LHS is exactly to build such a shape, and these extra efforts make our proof technically non-trivial.

Also, our formulas are evaluated at *pairs* of states, where each of the states can occur by itself (and, not just as a constituent of an ordered pair), which makes it possible for us to study the relationship between two states directly. In $\mathsf{K} \times^\delta \mathsf{K}$, even though formulas are evaluated at pairs of states, these pairs themselves form nodes in the domain. As a result, product logic cannot express

¹⁷ In literature, the notation $\mathsf{K} \times^\delta \mathsf{K}$ has different meanings: [31, 28] use it for a product logic augmenting $\mathsf{K} \times \mathsf{K}$ with δ , while [29] uses it for a product logic whose frame $(W \times W, R^r, R^l)$ is the product of a frame (W, R) with itself.

directly the more fine-grained relation (i.e., identity) between the two components forming a pair. In [31, 28], δ is interpreted as a special subset of the domain, not necessarily consisting of pairs formed by the same components from those dimensions. Therefore, we can say that constant \mathcal{I} works on a meta level while δ in [31, 28] is a notion of the object level. But definitely, this does not exclude a possible ‘mixture’ of the two frameworks. As suggested in Section 6, technically LHS can be reduced to product logics with δ . On the other hand, product models themselves can also be viewed as special models (with two relations) for LHS (and then \mathcal{I} denotes the identity of *two* pairs).

8 Conclusion and future work

Summary. This chapter explores the game of hide and seek from a modal logic perspective. We design a logical framework LHS allowing us to talk about moves for each player, as well as meeting of the players. Many properties of the logic and its fragment LHS^- (without the constant \mathcal{I}) are studied, for instance, their expressive powers, computational behaviors, axiomatization and the connections with other relevant traditions in the literature. For expressiveness, new notions of bisimulation are proposed, which together with the corresponding notions of first-order translation lead us to characterization theorems for the logics. Also, w.r.t. computational behavior, we show that the model checking problems for both the logics are P-complete; and perhaps surprisingly, the harmless-looking constant \mathcal{I} drastically increases the complexity of the satisfiability problem of LHS: LHS is proved to be undecidable while LHS^- is decidable. Finally, we establish other desired properties of LHS^- , including a complete Hilbert style proof system, and we look into the connections between our logic and the framework of product logics, which shed light on further study of LHS. As stated, although the logics are designed for the game with two players, all the results can be easily generalized for the settings with more players.

Further directions. In addition to the open problems that have been formulated along our discussion, there are other directions to be pursued. It makes sense to have a better understanding of the phenomenon of undecidability of LHS on account of the constant \mathcal{I} , possibly by investigating some alternative logics (e.g., the logic mentioned in Remark 2). In Sections 6 and 7, we have seen certain similarities and differences between our work and product logics, but many more issues remain to be explored in this direction. Moreover, there are many other variants of the game [14, 30, 39, 21], and it is meaningful to develop formal frameworks for studying strategic reasoning in these games.

It is also important to study the axiomatizability of LHS, for which it might be useful to analyze the techniques developed for $\mathbf{K} \times^\delta \mathbf{K}$ [29]. For expressiveness, besides the expressive power of LHS w.r.t. models, the equality constant \mathcal{I} improves the frame definability of LHS as well, and we leave the possibility to have a comprehensive understanding of this property for the future. Another direction is to develop the proof theory for our logics, for instance, provide sequent calculi and tableau systems for the same. Finally, it is worthwhile to generalize our

results to broader settings and consider the extensions of LHS and LHS^- with further operators, such as *graded modalities* [20, 44] to talk about the degrees of states in a graph.

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