Knowing is winning: An epistemic approach to the hide and seek game

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Abstract. The game of Hide and Seek serves as an important paradigm in computer science for studying search problems, and has attractive analogies with logic. This chapter focuses on an imperfect-information version of the game, where, as in realistic pursuit-evasion environments, players have only limited abilities. In the course of the play, they keep changing their positions and reason about each other's positions based on what they have already known. In this work, we will identify the game design and explore a logical framework, epistemic logic for the hide and seek game (ELHS), to reason about the game. For this purpose, it is crucial to have a formal tool that captures how players update their knowledge, enabling us to characterize key aspects of the game, including the players' winning positions. By exploring the connections between our logic and other existing logical systems, we show that the 'static fragment' of ELHS without the dynamic operators has a decidable satisfiability problem and a P-complete model-checking problem. We finish with a detailed analysis of such games using the logical language.

Keywords: Hide and seek · Modal logic · Computational complexity · Knowledge updates

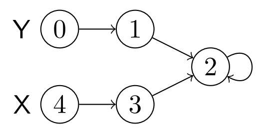
1 Introduction

The hide and seek game, as played on graphs, was previously explored by [11] from a logical perspective, where a modal language was introduced to describe the interactions between the two players. This language included a propositional constant for identity to represent the scenario where the seeker catches the hider. In this chapter, we diverge from the modeler's perspective adopted in [11] and instead concentrate on the perspectives of the players, introducing the uncertainties of players into the scenario. Consequently, an epistemic dimension is integrated into the existing frameworks that describe such games.

The work is centered on the reasoning abilities of the players—specifically, how they reason about the positions of themselves and their opponents by making use of the knowledge and information acquired during the gameplay. This reasoning ability is contingent upon their observational capabilities, referred to as the *sight* of the players. The dynamic interactions between the players are our primary focus, and we propose a formal framework to support the study and analysis of these interactions.

More formally, a hide and seek game is played by a seeker and a hider on a *finite directed* graph in which *every vertex has successors* (an assumption made for the ease of presentation), as illustrated in the following example:

Example 1. In the graph depicted below, seeker X (female) is at 4, and hider Y (male) at 0. They know the graph structure and their own positions. Let us assume that a player can always see the other player who is close to oneself: e.g., the player who is at the same position or at the vertices reachable from one's current positions via an arrow or its converse. More specifically, in this example, X and Y do not know the positions of each other. But since they know the graph structure, X knows that Y must be at 0, 1 or 2, and Y knows that X must be at 2, 3 or 4.



A player whose turn it is has to move along an arrow. Let the game begin and X act first. Seeker X can only move to 3, since 4 has only access to 3. After this, X can know that Y is not at 2, as otherwise she can see Y directly. So, X knows that Y is at 0 or 1. Y knows that X is not at 4 after the movement, since it has no predecessor. So, Y knows that X is at 3 or 2 now, which are successors of the previous possibilities, considered by Y, of the position of X. Afterward, Y can only move to 1. After this, Y can know that X is not at 2 and so X is at 3. Also, since X can observe 2 and knows that Y is not at 0, X now knows that Y is at 1 and wins.

Though simple, the above example already shows several intriguing aspects. First, their knowledge keeps changing during the play of the game. Next, at the final stage of the game, Y is not directly observed by X, but X can still get to know where Y is, depending on the assumptions on players' knowledge and observational power. Making these notions precise will shed light on our understanding of the action-information interplay in the game.

Outline. Section 2 lays out our basic assumptions about the hide and seek game explored in the chapter and discusses alternative ways to design it. Section 3 presents a logical framework to reason about the game and explores its properties. In Section 4, we show the applications of the logic by analysing a concrete game of hide and seek. Finally, Section 5 concludes the work and presents various further directions.

2 Basics of game design

Let us immediately give our basic assumptions regarding players' knowledge: (a) the graph structure is commonly known; (b) players also commonly know that there is a global clock and that an action in the game corresponds to a tick of the clock,[1] and more concretely, it is common knowledge that whose turn it is to move, but for instance, X may not know where Y moves from and/or where Y moves to; (c) players can remember what they have considered to be possible: when a player moves, the players infer the new possible situations from what they considered to be possible before that movement (for instance, before the movement of X in Example 1, Y considers it possible for X to be at 2, 3 or 4, and once X moves, he gets to know that X is at 3 or 2, which are successors of the previous possibilities).

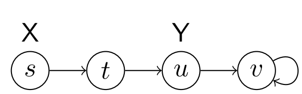
Actions: In each round, the seeker X moves first, and then the hider Y moves.[2]

Regarding the winning condition, we can easily stipulate that the seeker wins iff she moves to the same position as the hider, just like what [11, 12] did, using a propositional constant to express the meeting of two players. However, our exploration on the role of knowledge allows more possibilities to define the condition, and in what follows we will adopt a new one as follows:

Winning condition: Fix a natural number $n \in \mathbb{N}$. Seeker X wins if the position of the hider Y is *known* by X within n rounds, and she loses if she cannot know where the hider is after n rounds.

Ability of players: k-sight. To analyze the game from the players' perspectives, it is important to identify what players can know. There are two extreme cases about their *ability*: *full ability*, i.e., players are assumed to know the positions of each other at any stage of a game, and *no ability*, i.e., players do not have any ability to ensure they can know something, even their own positions. Inspired by [1, 9, 13], we introduce the notion of *k-sight of players* to describe their observational power: players can always see the players in the region that can be reached within k steps, via the arrows in two directions, from their own positions. For instance, X and Y in Example 1 have sight 1. Below is another example.

Example 2. In the picture below, assume again that both X and Y have sight 1.



[1] This is also referred to as *synchronicity* [10]. We wish to thank Davide Grossi for letting us notice the literature on the assumption.

[2] Similar to many other studies on this game, the version of the game considered in our work is turn-based (cf. e.g., [15, 7]). One can also consider any other specific order of their play, and even simultaneous play, but that will not affect our basic idea to analyze the game and the design of the logical tool in Section 3.

So, only s and t are in the sight of X and she knows that Y is not at s or t. However, after X moves to t, the position u of Y comes into the sight of X, so X knows where Y is.

It is important to note that the notion of sight is a tool to capture players' ability, but depending on the concrete situations players may be able to know more: for instance, in the case of 0-sight, although their ability can only enable players to know their own positions, they may still know more due to other reasons. Recall that in Example 1, X has 1-sight, but knows more due to her knowledge of the graph structure. But here, we hasten to emphasize that our goal in this work is to develop logical tools to track the changes in the knowledge of the positions of the players and to define the winning positions of the game. As a first step, we will only consider the setting with the knowledge of atomic facts and their Boolean combinations, but not higher-order knowledge (e.g., X knows that Y knows that X is at the vertex a or the vertex b). Although this may look restricted, the proposal developed in this way fits with many existing analyses for the imperfect information games of cops and robbers (see e.g., [7]), which are closely related to our game. We leave the work for the more intricate setting involving higher-order knowledge to another occasion.

We end this section by pointing out possible options regarding our assumptions of the game. For instance, (1) as stated earlier, graphs in this context are *serial*, but it is equally reasonable to consider graphs without any restrictions, (2) the players can act simultaneously, (3) different players may have different sights, (4) there can be more players with other kinds of ability, say, they can send each other messages, and (5) there may be other ways to define the winning condition, for instance, when there are $i > 1$ seekers, seekers may win when the position of the hider is their *distributed knowledge*. Some of these alternatives will be discussed in the later sections. For now, let us move to the details of the logical framework.

3 An epistemic logic for the hide and seek game

This section will present a logical framework that characterizes our assumptions about the game and enables us to reason about how players update their information during the gameplay.

3.1 Language and semantics

Inspired by the frameworks of [2, 3], we will use different *values* to encode different vertices in the graph of a game. Such values and the binary relation in a graph give us a semantic structure of first-order logic (FOL). Also, we will use *variables* to denote players, and then the current position of a player gives us the value of the corresponding variable. So, positions of all players can give us an *assignment* function σ that assigns values to variables, say, if player x is at a, then $\sigma(x) = a$.

We fix a *vocabulary* $Voc = (Pred, Cons, Var)$, where $Pred$ is a set of predicate symbols, containing a specific binary relation R describing the arrows of a game graph, $Cons$ is a non-empty, finite set of constants, and $Var = \{x, y\}$, meaning the players. As usual, elements of $Cons \cup Var$ are called *terms*. The language $\mathcal{L}$ for *the Epistemic Logic of the Hider and Seek game (ELHS)* is defined in the following:

Definition 1. *Formulas in* the language $\mathcal{L}$ for ELHS *are defined as follows:*

$$\alpha ::= P\boldsymbol{t} \mid t_1 \equiv t_2 \mid \neg\alpha \mid (\alpha \wedge \alpha)$$

$$\mathcal{L} \ni \varphi ::= \alpha \mid K_z t \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_z\alpha \mid [z]\varphi$$

where $t, t_1, t_2 \in Cons \cup Var$ *are terms,* $\boldsymbol{t}$ *is a tuple of terms,* $P \in Pred$ *is a predicate symbol, and* $z \in Var = \{x, y\}$ *is a variable. Other Boolean connectives* $\top, \bot, \to, \vee$ *are defined as usual. Also, for a non-empty set* $T \subseteq Cons \cup Var$*, we use* $K_z T$ *for* $\bigwedge_{t \in T} K_z t$.

As usual, $P\boldsymbol{t}$ and $t_1 \equiv t_2$ are *atomic formulas* (in particular, formula Rt_1t_2 states that *the value of* t_2 *is a successor of the value of* t_1), $K_z t$ reads *player* z *knows the value of* t, $K_z\varphi$ means z *knows that* φ, and $[z]\varphi$ expresses *after any movement of* z, φ *is the case.* It is important to notice that the language itself does not contain formulas for high order knowledge (e.g., $K_x K_y c$), and when $K_z\varphi$ is a well-defined formula, it must be the case that φ is atomic or a Boolean combination of atomic formulas: as we shall see, to define both the changes of knowledge for positions and the winning positions of the players in the game, we even do not need the formulas of the form $K_z\varphi$, but we add those $K_z\alpha$ to the language for convenience.

Definition 2. *A* model for ELHS *is a tuple* $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$*, where*

- $\mathbf{D}$ *is a non-empty, finite set of values (also called vertices or positions).*
- $\mathbf{I}$ *is the interpretation function such that*
  - *For each m-ary predicate symbol* $P \in Pred$, $\mathbf{I}(P) \subseteq \mathbf{D}^m$ *is an m-ary relation on* $\mathbf{D}$*. In particular,* $\mathbf{I}(R)$ *is a binary relation on* $\mathbf{D}$.
  - *For each* $c \in Cons$, $\mathbf{I}(c) \in \mathbf{D}$.
- $\Sigma \subseteq \mathbf{D}^{Var}$ *is a non-empty set of* situations *(of the players' positions).*
- *For each* $z \in \{x, y\}$, $\sim_z \subseteq \Sigma \times \Sigma$ *is an equivalence relation.*

When the interpretation function $\mathbf{I}$ is clear, we often write $\mathbf{P}$ for $\mathbf{I}(P)$. Intuitively, $(\mathbf{D}, \mathbf{R})$ represents a graph where a game is played;[3] Σ is a collection of possible situations of a game; and the relations $\sim_{z \in Var}$ are the indistinguishability relations of players (e.g., $\sigma_1 \sim_x \sigma_2$ means that player x cannot distinguish between situations σ_1 and σ_2).

The class of all models captures the case where players do not have any ability, not even the 0-sight: for instance, it might be the case that $\sigma_1 \sim_x \sigma_2$ and $\sigma_1(x) \neq \sigma_2(x)$, which means that x does not know where herself is. To capture the k-sight ability, we first define an auxiliary notion as follows:

[3] For simplicity, we do not have any restrictions on $\mathbf{R}$, and the previous assumption of seriality on $\mathbf{R}$ that we have taken for our game is a special case.

Definition 3. *Let $(\mathbf{D}, \mathbf{R})$ be a finite graph. For any $s \in \mathbf{D}$ and $m \in \mathbb{N}$, we inductively define the following:*

$$\mathbb{D}^0(s) := \{s\}$$
$$\mathbb{D}^{m+1}(s) := \mathbb{D}^m(s) \cup \{t \in \mathbf{D} : \exists u \in \mathbb{D}^m(s) \textit{ s.t. } (u,t) \in \mathbf{R} \textit{ or } (t,u) \in \mathbf{R}\}.$$

By the definition, $t \in \mathbb{D}^k(s)$ intuitively states that the 'distance' of s and t is not more than k, and more precisely, the vertex t can be reached from s within k steps via the symmetric closure of $\mathbf{R}$. With the help of this, we introduce the following notion:

Definition 4. *A* k-sight model *is a model $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ such that for each $z \in Var$ and $\sigma, \sigma' \in \Sigma$, if $\sigma \sim_z \sigma'$, then for all $z' \in Var$ s.t. $\sigma(z') \in \mathbb{D}^k(\sigma(z))$, it holds that $\sigma(z') = \sigma'(z')$.*

One can check that in a k-sight model, if the distance between the position of a player z and the position of player z' is not more than k, then in all situations that cannot be distinguished by z, player z' always has the same position, which means that z knows the position of z'. This characterizes the k-sight ability. In what follows, we work with k-sight models, and ELHS is defined based on them.

Remark 1. The notion of k-sight models can be adapted to capture cases that players have different sights. For each $z \in \{x, y\}$, we use k_z for the sight of the player. Then, to characterize this more complicated situation, we just need to replace the restriction imposed on k-sight models with the following:

For each $z \in Var$ and $\sigma, \sigma' \in \Sigma$, if $\sigma \sim_z \sigma'$, then for all $z' \in Var$ such that $\sigma(z') \in \mathbb{D}^{k_z}(\sigma(z))$, $\sigma(z') = \sigma'(z')$.

In what follows, given a model $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ and a situation $\sigma \in \Sigma$, for any term t, we use $t^{(\mathbf{I},\sigma)}$ for the value of the term: if $t \in Cons$, then $t^{(\mathbf{I},\sigma)} := \mathbf{I}(t)$; and if $t \in Var$, then $t^{(\mathbf{I},\sigma)} := \sigma(t)$.

Now we move to presenting the semantics for the logic. Truth conditions for the fragment of $\mathcal{L}$ without operators $[z]\varphi$ are straightforward, and key clauses are as follows:

$$\begin{aligned}
M, \sigma \vDash P(t_1, \ldots, t_n) \quad & \text{iff} \quad (t_1^{(\mathbf{I},\sigma)}, \ldots, t_n^{(\mathbf{I},\sigma)}) \in \mathbf{I}(P) \\
M, \sigma \vDash t_1 \equiv t_2 \quad & \text{iff} \quad t_1^{(\mathbf{I},\sigma)} = t_2^{(\mathbf{I},\sigma)} \\
M, \sigma \vDash K_z t \quad & \text{iff} \quad \text{for all } \sigma' \in \Sigma, \text{ if } \sigma' \sim_z \sigma, \text{ then } t^{(\mathbf{I},\sigma)} = t^{(\mathbf{I},\sigma')} \\
M, \sigma \vDash K_z \varphi \quad & \text{iff} \quad \text{for all } \sigma' \in \Sigma, \text{ if } \sigma' \sim_z \sigma, \text{ then } M, \sigma' \vDash \varphi
\end{aligned}$$

It remains to show the truth condition for the movement operators $[z]\varphi$. In terms of the game, a desired clause would give us an automated update mechanism to capture the effect of players' actions, which is involved with both the changes of positions and the updates of the knowledge (of both the players). Let us first define binary relations $\mathsf{R}^{z \in Var}$ on all situations $\mathbf{D}^{Var}$:

For any $\sigma, \sigma' \in \mathbf{D}^{Var}$, we write $\mathsf{R}^z \sigma\sigma'$ if $(\sigma(z), \sigma'(z)) \in \mathbf{R}$ and for $z' \in Var \setminus \{z\}$, $\sigma(z') = \sigma'(z')$.

Therefore, when $\mathsf{R}^z\sigma\sigma'$, the only difference of σ and σ' concerns the values of z: the value of z in σ' is an $\mathbf{R}$-successor of the value of z in σ, which intuitively describes the fact that after z moves from $\sigma(z)$ to $\sigma'(z)$, the situation σ becomes σ'. Given a class $\Sigma \subseteq \mathbf{D}^{Var}$ of situations and a variable z, we define

$$\mathsf{R}^z(\Sigma) := \{\sigma' \mid \textit{there is } \sigma \in \Sigma \textit{ s.t. } \mathsf{R}^z\sigma\sigma'\}.$$

When Σ is a singleton $\{\sigma\}$, we write $\mathsf{R}^z(\sigma)$ for $\mathsf{R}^z(\{\sigma\})$.

Given a model $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ and $\sigma \in \Sigma$, we use $\Sigma|\sigma$ to mean the subset $\Sigma' = \{\sigma' \in \Sigma \mid \sigma \sim_z \sigma' \textit{ for some } z \in Var\}$, which excludes the states that are 'irrelevant' to the current situation σ.

Now we can show the truth condition for the movement operators $[z]\varphi$:[4]

Definition 5. *For the case that both the players have the same sight k, the truth condition for $[z]\varphi$ is as follows:*

$$(\mathbf{D}, \mathbf{I}, \Sigma, \sim), \sigma_1 \vDash [z]\varphi \quad \textit{iff} \quad \textit{for all } \sigma_2 \in \mathsf{R}^z(\sigma_1),\ (\mathbf{D}, \mathbf{I}, \Sigma', \sim'), \sigma_2 \vDash \varphi,$$

where the new Σ' is given by the following:

(a) if $\sigma_2(x) \in \mathbb{D}^k(\sigma_2(y))$, then $\Sigma' = \{\sigma_2\}$,
(b) if $\sigma_2(x) \notin \mathbb{D}^k(\sigma_2(y))$, then $\Sigma' = \{\sigma' \in \mathsf{R}^z(\Sigma|\sigma_1) \mid \sigma'(x) \notin \mathbb{D}^k(\sigma'(y))\}$,

and the new relations $\sim'_{z\in\{x,y\}}$ on Σ' are obtained by the following:

(c) $\sigma_1 \sim'_z \sigma_2$ iff $\sigma_1(z) = \sigma_2(z)$.

In the definition above, the new class Σ' is given by the clauses (a) and (b), and the new indistinguishability is given by the clause (c).

The clause (a) aims to deal with the case that after the movement, the two players are in the sight of each other. In this case, both of them know the actual case and they would not consider other situations to be possible. Moreover, the clause (b) tackles the case that after the movement, the two players are not in the sight of each other. Different from that of (a), players only consider the situations in which they are not in the sight of each other to be possible.

With the definition above, one can see that the class Σ' depends on the new situation σ_2, and different σ_2 can give us different Σ'. But it always holds that $\Sigma' \subseteq \mathsf{R}^z(\Sigma)$ and $\sigma_2 \in \Sigma'$. The definitions for the new Σ' and $\sim'$ show that after the movement of z, each player z' reasons about the new situations, based on the k-sight ability, with what z' considered to be possible before the movement.

When M is a k-sight model, the resulting model obtained by the updating M in the manner described above is again a k-sight model. We will provide some examples for the updates further in Section 4.

Remark 2. As stated, it is also interesting to consider the case that x and y act simultaneously. To capture this, we are going to define 'group movement operators' $[Var]\varphi$. Similar to the case of R^z that characterizes the changes of the situations caused by a movement of z, let us define R^{Var} such that

[4] The authors would like to thank Alexandru Baltag for a very useful discussion on the notion of the update.

$$\text{for any } \sigma, \sigma' \in \mathbf{D}^{Var},\ \mathsf{R}^{Var}\sigma\sigma' \text{ iff for any } z \in Var,\ (\sigma(z), \sigma'(z)) \in \mathbf{R},$$

which captures the changes of situations caused by the simultaneous movements of both x and y. For a set $\Sigma \subseteq \mathbf{D}^{Var}$, we can define

$$\mathsf{R}^{Var}(\Sigma) := \{\sigma' \mid \textit{there is } \sigma \in \Sigma \textit{ s.t. } \mathsf{R}^{Var}\sigma\sigma'\},$$

and again, when Σ is a singleton $\{\sigma\}$, we write $\mathsf{R}^{Var}(\sigma)$ for $\mathsf{R}^{Var}(\{\sigma\})$. Now, the truth condition for $[Var]\varphi$ is given by the following:

$$(\mathbf{D}, \mathbf{I}, \Sigma, \sim), \sigma_1 \vDash [Var]\varphi \quad \text{iff} \quad \text{for all } \sigma_2 \in \mathsf{R}^{Var}(\sigma_1),\ (\mathbf{D}, \mathbf{I}, \Sigma', \sim'), \sigma_2 \vDash \varphi,$$

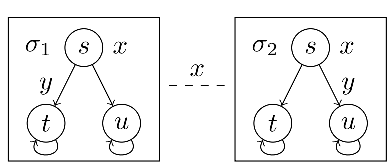
where the new class Σ' of situations and the indistinguishability relations $\sim'$ are given by the same clauses as that in Definition 5, except we now need to use R^{Var} instead of R^z.

To conclude this section, it is important to acknowledge the well-established field that focuses on the dynamics of knowledge, namely *dynamic-epistemic logic* (DEL) (refer to [4, 8, 6] for examples). To understand how players reason within the hide and seek game, it is conceivable that the techniques developed for DEL, such as the product update from [4], could be applicable after appropriate modifications. Some valuable ideas for this are offered in [5], which uses the graph as an important parameter to construct *event models* that together with the epistemic models in our style can give rise to the product models for capturing the informational aspect after a movement. Although some kind of equivalence between this approach and our framework can still be established, in general the product method usually introduces some irrelevant states to the models. Compared with our approach, those product models often have a much larger size, and we believe that ELHS can be seen as a more succinct formalism. Instead of giving a formal comparison, in the remaining sections our discussion will center on ELHS, leaving the exploration of DEL's applicability and its precise connections with our framework for future research.

3.2 Properties of ELHS

Having seen the basics of the logic, we now move to exploring some of its properties, involving validities and computational behavior. Let us start with the former and the notions of *satisfiability* and *validity* are defined as usual. Due to the distinction between the logical designs, many valid schemata of FOL fail in this new setting. For instance, formula $t_1 \equiv t_2 \to (\varphi \leftrightarrow \varphi[t_1/t_2])$ is *not* a validity of ELHS, where $\varphi[t_1/t_2]$ is obtained by replacing t_2 in φ with t_1. To see this, we consider a counterexample:

Example 3. Consider the following model and for simplicity, assume that the sight k is 0.



As depicted, $\sigma_1(x) = \sigma_2(x) = s$, $\sigma_1(y) = t$, and $\sigma_2(y) = u$. Also, player y can distinguish between the two situations, but x cannot. (We omit the reflexive links for indistinguishability relations in the picture.) Also, let $c \in Cons$ and $\mathbf{I}$ be an interpretation function such that $\mathbf{I}(c) = t$. So, when σ_1 is the case, formula $c \equiv y \wedge K_x c \wedge \neg K_x y$ is true.

However, some restricted forms of the schema above still hold. We will show some validities of the logic, and before this, let us first introduce some abbreviations. For any natural number $n \in \mathbb{N}$ and terms t_1, t_2, we define the following expressing the distance between t_1 and t_2:

$$\mathsf{D}^0 t_1 t_2 := t_1 \equiv t_2$$
$$\mathsf{D}^{n+1} t_1 t_2 := \mathsf{D}^n t_1 t_2 \vee \bigvee_{t \in Var \cup Cons} (\mathsf{D}^n t_1 t \wedge (Rtt_2 \vee Rt_2 t))$$

Since $Var \cup Cons$ is finite, the formulas above are well-defined. Now formula $\mathsf{D}^k zt$ means that the position of t is in the sight of z, which is useful in expressing the k-sight ability of the players.

Fact 1 *These formulas are valid w.r.t. k-sight models:*

(1) $t_1 \equiv t_2 \to (P\boldsymbol{t} \leftrightarrow P\boldsymbol{t}[t_1/t_2])$
(2) $K_z t_1 \equiv t_2 \to (K_z t_1 \leftrightarrow K_z t_2)$, *given that* $z \in Var$.
(3) $\mathsf{D}^k zt \to K_z t$, *given that* $z \in Var$.
(4) $(\mathsf{D}^k zt_1 \wedge t_1 \equiv t_2) \to K_z t_1 \equiv t_2$, *given that* $z \in Var$.
(5) $(K_z T \wedge P(t_1, t_2, \ldots, t_m)) \to K_z P(t_1, t_2, \ldots, t_m)$, *given that* $\{t_1, \ldots, t_m\} \subseteq T$.

Proof. Let $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ be a k-sight model and $\sigma_1 \in \Sigma$. The validity of formula (1) is easy to see. Let us now consider others.

For formula (2), assume that $M, \sigma_1 \vDash K_z t_1 \equiv t_2 \wedge K_z t_1$. Then, for each situation σ_2 with $\sigma_1 \sim_z \sigma_2$, it holds that $\sigma_2(t_1) = \sigma_2(t_2)$ and $\sigma_2(t_1) = \sigma_1(t_1)$, so $\sigma_2(t_2) = \sigma_1(t_2)$. Thus, $M, \sigma_1 \vDash K_z t_2$.

The validity of formula (3) holds directly by the k-sight restriction.

For principle (4), assume $M, \sigma_1 \vDash \mathsf{D}^k zt_1 \wedge t_1 \equiv t_2$. Then, both t_1 and t_2 are in the sight of z. So, z knows the values of t_1 and t_2, which then ensures that z knows the fact that $t_1 \equiv t_2$.

Finally, we turn to (5). Assume that $K_z T \wedge P(t_1, t_2, \ldots, t_m)$ is true at σ_1. We suppose for reductio that $M, \sigma_1 \nvDash K_z P(t_1, t_2, \ldots, t_m)$. Then, there is some σ_2 s.t. $\sigma_1 \sim_z \sigma_2$ and $(t_1^{(\sigma_2, \mathbf{I})}, \ldots, t_m^{(\sigma_2, \mathbf{I})}) \notin \mathbf{P}$. However, since $K_z T$, we know that for each $1 \leq n \leq m$, $\sigma_1(t_n) = \sigma_2(t_n)$, which can give us $(t_1^{(\sigma_2, \mathbf{I})}, \ldots, t_m^{(\sigma_2, \mathbf{I})}) \in \mathbf{P}$, a contradiction. This completes the proof. □

Many of these principles state facts about the game. Principle (1) is a restricted case in which we can substitute a term with another when they have the same value. Formulas (2) and (3) concern the k-sight ability: (2) states that when z knows that t_1 and t_2 have the same value, then z knows the value of one of them just in the case that z knows the value of the other, and (3) means if t is in the sight of z, then z knows where t is. (4) states that if t_1 is in the sight of z and t_1 has the same value as t_2, then z knows that they have the same value. Finally, validity (5) indicates that knowing the values of T means knowing the atomic facts involving T: when P is R, it characterizes the assumption that *players know the graph of a game*.

W.r.t. ELHS, many of its meta-properties deserve to be explored. For instance, it is worthwhile to study the complexity of its model-checking problem and satisfiability problem. We have the following:

Theorem 1. *The model-checking problem for the fragment of ELHS without* $[z]$ *is P-complete.*

Proof. To show this, we first show that we can embed the propositional logic into ELHS, which is P-complete and provides us with a lower bound, and then show that ELHS can be embedded into the fragment of FOL with a fixed number of variables, which is also P-complete [16] and offers an upper bound. As we shall see, both the translation functions are polynominal in the size of input formulas.

Now let us begin. First of all, let us define a syntactic function T from the propositional logic PL to ELHS. Fix a constant $c \in Cons$. Details are as follows:

$$\mathsf{T}(p) := Pc$$
$$\mathsf{T}(\neg\varphi) := \neg\mathsf{T}(\varphi)$$
$$\mathsf{T}(\varphi \wedge \psi) := \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi)$$

Let $\mathcal{V}$ be a valuation for the propositional logic. We can construct a model $M_{\mathcal{V}} = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ for ELHS as follows:

- $\mathbf{D} := \{s\}$.
- $\Sigma := \{\sigma\}$, where $\sigma(x) = \sigma(y) = s$.
- For any $z \in Var$, $\sim_z := \{(\sigma, \sigma)\}$.
- $\mathbf{I}$ is given by the following:
  - For any $c' \in Cons$, $\mathbf{I}(c') := s$.
  - For any unary predicate symbol P_i, $\mathbf{I}(P_i) := \{s\}$ if $\mathcal{V}(p_i) = 1$, otherwise $\mathbf{I}(P_i) := \emptyset$. For any other predicate symbol Q, $\mathbf{I}(Q) := \emptyset$.

Now we can show that for any formula φ of the propositional logic, we always have the following equivalence:

$$\mathcal{V} \vDash \varphi \quad \text{iff} \quad M_{\mathcal{V}}, \sigma \vDash \mathsf{T}(\varphi).$$

The proof is by induction on propositional formulas, and we skip the details.

For an upper bound, we can embed the fragment into a fragment of FOL with a fixed number of variables. In the first order language, (a) we use *two new variables* u, v to denote situations Σ, where $u, v \notin Var$; (b) we have new functional symbols $\{f(u,t), f(v,t) \mid t \in Var \cup Cons\}$, intuitively meaning the values of t given by situations u and v; (c) for each $z \in Var$, the first-order language contains a binary relation E^z among situations (for instance, $E^x uv$

expresses that x cannot distinguish between situations u and v); and (d) the language contains a new unary predicate symbol S such that Su intuitively means u is a situation in our models. Clauses of the translation $\mathcal{T}_u$ from the fragment of $\mathcal{L}$ into the first-order language are as follows:

$$\begin{aligned}
\mathcal{T}_u(P(t_1,\ldots,t_m)) &:= P(f(u,t_1),\ldots,f(u,t_m))\\
\mathcal{T}_u(\neg\varphi) &:= \neg\mathcal{T}_u(\varphi)\\
\mathcal{T}_u(\varphi\wedge\psi) &:= \mathcal{T}_u(\varphi)\wedge\mathcal{T}_u(\psi)\\
\mathcal{T}_u(K_z t) &:= \forall v(Sv\wedge E^z uv\to f(u,t)\equiv f(v,t))\\
\mathcal{T}_u(K_z\varphi) &:= \forall v(Sv\wedge E^z uv\to \mathcal{T}_v(\varphi))
\end{aligned}$$

Also, given a model $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$, we now proceed to construct a model $\mathfrak{M} = (D, I)$ as follows:

- $D := \mathbf{D} \cup \Sigma$
- For any $z \in Var$, $\sigma^+(z) = \sigma(z)$, and $\sigma^+(u) = \sigma^+(v) = \sigma$
- The interpretation for constant symbols $Cons$ and the old predicate symbols $Pred$ are the same as $\mathbf{I}$. Also, we define the following:
  - $I(S) := \Sigma$
  - Given an assignment σ^+,
    $(f(c,u))^{(I,\sigma^+)} := \mathbf{I}(c)$ and $(f(z,u))^{(I,\sigma^+)} := \sigma^+(u)(z)$.
  - For any $z \in Var$, $I(E^z) := \{(\sigma_1,\sigma_2) \mid \sigma_1 \sim_z \sigma_2\}$.

For any assignment $\sigma \in \Sigma$, we write σ^+ for an arbitrary assignment of the extended set of variables such that for any $z \in Var$, $\sigma^+(z) = \sigma(z)$. Now we can prove the following equivalence:

$$M,\sigma \vDash \varphi \quad \text{iff} \quad \mathfrak{M},\sigma^+[u:=\sigma] \vDash \mathcal{T}_u(\varphi),$$

where for any $s \in Var \cup \{v\}$, $\sigma^+[u:=\sigma](s) = \sigma^+(s)$ and $\sigma^+[u:=\sigma](u) = \sigma$. We leave the details of the proof to the reader. Now, as proved in [16], the model-checking problem for FOL with a fixed number of variables is P. This completes the proof. □

Theorem 2. *The satisfiability problem for the fragment of ELHS without operators* $[z]$ *is decidable.*

Proof. It can be proved by embedding the fragment without $[z]$ into *the logic of epistemic dependency* developed in [2]. Here we just state some the basic ideas of the translation of the language without $[z]$. One point is the different usages of variables: in our setting, variables intuitively denote agents (e.g., $K_x y$ expressing that x *knows where* y *is*), while agents in [2] are denoted by indexes of operators (e.g., $K_a y$ expressing that *agent* a *knows the value of* y, where a is neither a constant nor a variable). In a desired translation, we can associate each variable z in our language with a symbol a_z for agent. Another point is about the k-sight ability of agents, and we need to ensure that the property holds.

We denote by T the translation, and its clauses are as follows:

$$T(P\boldsymbol{t}) := \bigwedge_{z_1,z_2\in Var}(\mathsf{D}^k z_1z_2 \to K_{a_{z_1}}z_2) \wedge P\boldsymbol{t}$$
$$T(t_1\equiv t_2) := \bigwedge_{z_1,z_2\in Var}(\mathsf{D}^k z_1z_2 \to K_{a_{z_1}}z_2) \wedge t_1\equiv t_2$$
$$T(\neg\varphi) := \bigwedge_{u,v\in Var}(\mathsf{D}^k uv \to K_{a_u}v) \wedge \neg T(\varphi)$$
$$T(\varphi\wedge\psi) := \bigwedge_{u,v\in Var}(\mathsf{D}^k uv \to K_{a_u}v) \wedge T(\varphi)\wedge T(\psi)$$
$$T(K_z t) := \bigwedge_{u,v,s\in Var}K_{a_s}(\mathsf{D}^k uv \to K_{a_u}v) \wedge K_{a_z}t$$
$$T(K_z \varphi) := \bigwedge_{u,v,s\in Var}K_{a_s}(\mathsf{D}^k uv \to K_{a_u}v) \wedge K_{a_z}T(\varphi)$$

As we can see, expressions for high order knowledge are involved in the translation, which are allowed in the logic of epistemic dependency. To understand the translation, we present some examples. First, one can infer that $T(\neg P\boldsymbol{t})$ is equivalent to

$$\bigwedge_{u,v\in Var}(\mathsf{D}^k uv \to K_{a_u}v) \wedge \neg P\boldsymbol{t}.$$

Also, $T(\neg K_x t)$ is equivalent to

$$\bigwedge_{u,v\in Var}K_{a_s}(\mathsf{D}^k uv \to K_{a_u}v) \wedge \neg K_{a_x}t.$$

Finally, $T(\neg K_x Pt)$ is equivalent to

$$\bigwedge_{u,v,s\in Var}K_{a_s}(\mathsf{D}^k uv \to K_{a_u}v) \wedge \neg K_{a_x}Pt.$$

Moreover, the logic of epistemic dependency has the finite model property [2]. For every $T(\varphi)$, if the formula is satisfiable in a finite model M of the logic of epistemic dependency, then we can obtain a k-sight model for φ from M. This completes the proof. □

However, when the operators $[z]$ are taken into consideration, the cases become more involved (cf. Section 4) which need further considerations w.r.t. the computational properties of the full ELHS.

Conjecture: The satisfiability problem for ELHS is decidable, and its model-checking problem is in PSPACE.

We leave it for future work. For now, let us end this section by a discussion on expressing those relations R^x:

Extensions of the language. Given a model M, relations $\mathsf{R}^{z \in Var}$ might not be described by our language. But we can extend our vocabulary and M s.t. $\mathsf{R}^z \sigma_1 \sigma_2$ iff there are some $c_1, c_2 \in Cons$ such that for the other variable $z' \neq z$, $z' \equiv c_1$ is true at both σ_1 and σ_2, and Rzc_2 and $z \equiv c_2$ are true at situations σ_1 and σ_2, respectively. Moreover, working only with the k-sight models satisfying this property does not affect ELHS, in the sense of the following:

Theorem 3. *If a formula $\varphi \in \mathcal{L}$ is satisfiable w.r.t. (k-sight) models $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$, then it is satisfiable with regard to the class of (k-sight) models $M^+ = (\mathbf{D}, \mathbf{I}^+, \Sigma, \sim)$ s.t. whenever $\mathsf{R}^z \sigma_1 \sigma_2$, there are constants c_1, c_2 s.t. for the other variable z', $\mathbf{I}^+(c_1) = \sigma_1(z') = \sigma_2(z')$, $(\sigma_1(z), \mathbf{I}^+(c_2)) \in \mathbf{R}$, $\sigma_2(z) = \mathbf{I}^+(c_2)$.*

Proof. We first show how to construct a model from a model that may not satisfy the property, and then show that the construction preserves the truth of formulas.

There are several ways to achieve the first goal. One of them is to name all vertices with constants. Let $M = (\mathbf{D}, \mathbf{P}, \Sigma, \sim)$ be a model, and $Cons' := \{c_s \mid s \in \mathbf{D}\}$ be a set of constants disjoint from $Cons$. We define $M^+ = (\mathbf{D}, \mathbf{I}^+, \Sigma, \sim)$ as follows:

- For each predicate symbol $P \in Pred$, $\mathbf{I}^+ := \mathbf{I}$.
- For each $c \in Cons$, $\mathbf{I}^+(c) := \mathbf{I}(c)$, and for each $c_s \in Cons'$, $\mathbf{I}^+(c_s) := s$.

Now, we prove that for each φ built from the vocabulary with $Cons$, we have

$$M, \sigma \vDash \varphi \text{ iff } M^+, \sigma \vDash \varphi.$$

The proof proceeds by induction on φ. Instead of showing all the details, let us briefly comment on the case for $[z]\varphi$. To see that the conclusion also holds for this, it is important to see that the model $M_{[z]}$, obtained by updating M' with the dynamic operator $[z]$, also extends $M_{[z]}$ that is obtained by updating M with $[z]$ with the interpretations for the new constants $\{c_s \mid s \in \mathbf{D}\}$, so the induction hypothesis applies. We leave the details to the reader. □

The result concerns the technical aspect of logic which helps us in expressing the interplay between the movements of players and the changes of situations in the hide and seek game in terms of the language of ELHS. We believe that it is useful in many aspects, including the exploration of a Hilbert-style proof system for the logic. Also, it enables us to capture the game in a more precise way. For instance, previously the truth of the formula $\mathsf{D}^k xy$ is just a sufficient condition for the case that x and y are in the sight of each other, while now with the extension of the language, we can easily make it both sufficient and necessary for the case.

4 Applications: Analyzing games

Now we apply the formal framework to the games. We will use logic to track players' knowledge, to describe the winning conditions for players, among others. In this section, we assume that the round-restriction $n \in \mathbb{N}$ imposed on the winning condition is big enough.

Linking logic and games more tightly. It is not hard to understand that when a player $z \in \{x, y\}$ moves, z can know more, while the other player may learn or lose the information about the position of z. Our update mechanism for $[z]$ enables us to analyze the complete progress of a hide and seek game. But, it is important to realize that *not* all k-sight models together with their $[z]$-updates capture a hide and seek game: we need to impose necessary restrictions on models.

Definition 6. *Let $\mathbb{G}$ be a hide and seek game with a seeker x and a hider y. By σ_0 we denote the original positions of players. We say that* a k-sight model $M = (\mathbf{D}, \mathbf{I}, \Sigma, \sim)$ matches $\mathbb{G}$*, if the following holds:*

(1) $(\mathbf{D}, \mathbf{R})$ *is the playground of* $\mathbb{G}$.
(2) *For any $\sigma \in \mathbf{D}^{Var}$, $\sigma \in \Sigma$ iff there is some player cannot distinguish between σ_0 and σ.*
(3) *For any $z \in Var$ and $\sigma, \sigma' \in \Sigma$, $\sigma \sim_z \sigma'$ if, and only if,*
  (3.1) *Epistemically, z cannot distinguish between σ and σ', and*
  (3.2) *σ is a situation in which the seeker wins iff σ is also such a situation.*

The clauses ensure that the players' initial knowledge before the game is described in a correct way. The condition (3.2) is an additional requirement here, which is not a 'purely epistemic' clause. In contrast, it is a restriction imposed by the *termination-requirement* of the game: once some player wins, the game ends and all players know this. More generally, it is not just a condition for the initial model given in Definition 6, but also an extra policy regulating how we have further updates: in what follows, we extend the clause (c) in Definition 5 to the following:

$$\sigma_1 \sim'_z \sigma_2 \quad \text{iff} \quad \sigma_1(z) = \sigma_2(z), \text{ and } \sigma_1 \text{ is a situation that the seeker wins iff } \sigma_2 \text{ is also such a situation } (\mathsf{TR})$$

Now let us use the formal tool to analyze the game in Example 1.

A concrete game of hide and seek. We will write x and y for the seeker and the hider respectively. Recall that they have the sight 1. Moreover, in each round, x acts first. We denote by σ_0 the initial situation. In the remainder of the example, we use tuples of values to specify the positions of x, y (in this order), to denote situations. For instance, σ_0 is written as $(4, 0)$. For simplicity, we highlight the actual situation with an underline.

Based on the 1-sight ability, x knows her own position. W.r.t. the position of the hider y, x knows that y is not at 4 or 3, but at 0, 1 or 2. So, for player x, all of the following situations are possible:

$$\boxed{x: \quad \underline{(4,0)},\ (4,1),\ (4,2)}$$

Also, y knows where himself is, but he cannot distinguish between the following:

$$\boxed{y: \quad \underline{(4,0)},\ (3,0),\ (2,0)}$$

All these aforementioned situations together form the set Σ_0 of possible situations of any model matching the game. Also, the indistinguishability relations $\sim_x$ and $\sim_y$ among these situations are as described above.

Now, players begin to act. Player x has only one option:

x moves to 3. The actual case becomes $(3,0)$, written σ_1. Let us analyze the effects of the action on knowledge of positions.

First of all, since they are not in the sight of each other, by the definition of the update, we know that

$$\Sigma_1 = \{\sigma \in \mathsf{R}^x(\Sigma_0|\sigma_0) \mid \sigma(x) \notin \mathbb{D}^1(\sigma(y))\} = \{(3,0),(2,0),(3,1)\}.$$

Moreover, by the definition, the indistinguishability relations among the situations are as follows:

$$\boxed{x: \quad \underline{(3,0)},\ (3,1)} \qquad \boxed{y: \quad \underline{(3,0)},\ (2,0)}$$

Recall the situations that cannot be distinguished by x in the round 0. She now does not consider the case that y is at 2 to be possible: in the actual case $(3,0)$ she cannot observe directly y, but $(3,2)$ is a case in which x and y are in the sight of each other.

Besides, for y, the case that $(4,0)$ is impossible now, since 4 is not a successor of any vertex that were considered to be possible by y, and more generally, we have $(4,0) \notin \mathsf{R}^x(\Sigma_0)$.

Remark 3. It is worthy to note that taking the temporal dimension into account would be useful in spelling out more details. For instance, at the current stage of the example, y considers it is possible that x is at 2, which is a successor of both 3 and 2. The two states are successors of $4,3,2$, which were the positions of x that were considered to be possible by y before the movement of x. However, after the movement, y would realize that before the movement of x, it could not be the case that x was at 3, as otherwise y loses immediately after the movement of x (from 3 to 2), and the winning of x in such a new setting would make y know that $(2,0)$ is the actual case (recall the clause (TR)).

Let us now move to the next stage:

y moves to the only option 1. After this, it is again the case that x and y are not in the sight of each other. The new class Σ_2 of situations given by our clauses for update is as follows:

$$\Sigma_2 = \{\sigma \in \mathsf{R}^y(\Sigma_1|\sigma_1) \mid \sigma(x) \notin \mathbb{D}^1(\sigma(y))\} = \{(3,1)\}.$$

Now both the players know the actual situation and the positions of each other. It is meaningful to emphasize that although Σ_2 is a singleton set, it is not obtained by the clause (a) in Definition 5, but obtained by the clause (b): this indicates that the realization of the actual situation depends on their indirect reasoning (with the knowledge about the graph structure and the k-sight ability), but not on the direct observation. This completes the analyses of the example.

In addition, the language can be used to verify whether a player has a winning strategy. For instance, when the round-restriction n is 2, the seeker has a winning strategy iff the following is true at the initial situation:

$$K_x y \vee \langle x\rangle K_x y \vee \langle x\rangle[y]K_x y \vee \langle x\rangle[y]\langle x\rangle K_x y \vee \langle x\rangle[y]\langle x\rangle[y]K_x y,$$

where $\langle x\rangle$ is the dual operator of $[x]$. With this, it is not hard to see another potential application of the logic to the game: the model-checking problem for the formula of ELHS corresponds to the verification of existence of winning strategies of the players, and a complexity study would provide us an upper bound of the complexity of deciding this imperfect information game.[5]

To end this section, it is worth noting that how players update their knowledge depends on the following: (a) the graph, players' initial positions and sight, which together determine their initial knowledge of the positions of each other; (b) players' knowledge at the previous stage before the latest action takes place: a player z infers the new possibilities from the old ones; and (c) whether or not the game ends. These are captured by clauses in Definition 6 and (TR).

5 Conclusion and future work

Summary. In order to study the hide and seek game, this chapter provides a formal framework ELHS to capture the key notions and to describe players' reasoning about knowledge and actions. The basic properties of ELHS are studied. As we have seen, the framework ELHS is distinct from FOL whose valid schemata may fail here, and the computational behavior of the static fragment of ELHS is also explored. Certain validities of ELHS precisely capture our assumptions about the game. The information update mechanism is defined by a new dynamic operator and it is further employed in a concrete game setting to illustrate how it works. This is a first of its kind study of players' reasoning in hide and seek game, and we believe that our approach can be applied to other graph games too. These different viewpoints presented in this work will facilitate us to explore further properties of the formal language, as they enable us to use techniques developed in various areas.

Future work. In addition to what has been studied, a few further directions are deserving to be explored. For the game of hide and seek, there is a great deal of literature on the complexity studies of different versions of the game, exemplified by the 'cops and robbers' game [7], and we intend to do the same for the one introduced in our work and its different variants, and in process design efficient algorithms to construct winning strategies of players with respect to the different variants of the game. On the logic side, an immediate direction is to provide

[5] For a perfect-information version, see [12].

a complete Hilbert-style proof system for ELHS. Another important direction is to study the expressive power of ELHS, its extensions for the variants of the game, and the complexity of the model-checking and satisfiability problem of the full ELHS. Moreover, as stated, it is also feasible to develop logical frameworks using the techniques of dynamic-epistemic logic, and we leave a more detailed comparison between the two approaches to another occasion. Finally, it is instructive to notice that it is also interesting to consider the setting involving higher-order knowledge, in particular the more realistic case that players have restricted ability to reason about the knowledge of each other [14], which calls for a more sophisticated mechanism to capture the update of knowledge.

Acknowledgements. The logic of the hide and seek game was presented at the workshop 'Exploring Baltag's Universe' in January 2022. It was Alexandru Baltag's question that inspired the current research. We would like to thank Alexandru Baltag, Johan van Benthem and Davide Grossi for their useful comments for improvements. Sujata Ghosh acknowledges the Department of Science and Technology, Government of India for financial support with Reference No DST/CSRI/2018/202 under Cognitive Science Research Initiative (CSRI) to carry out this work. Fenrong Liu is supported by the Tsinghua University Initiative Scientific Research Program. Dazhu Li is supported by the National Social Science Foundation of China [22CZX063].

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