Cops only need factual knowledge to catch robbers

Dazhu Li[1,2] , **Sujata Ghosh**[3] ,
[1] Institute of Philosophy, Chinese Academy of Sciences
[2]Department of Philosophy, University of Chinese Academy of Sciences, Beijing China
[3]Indian Statistical Institute, Chennai, India
lidazhu@ucas.ac.cn, sujata@isichennai.res.in

Abstract

The cops and robber game is a well-studied model for investigating pursuit-evasion phenomena, among many others. Although many variants of this game have been studied in the literature, the epistemic assumptions underlying their game designs are often left implicit. In this work, we focus on the imperfect information version of the game and re-examine it from an epistemic perspective to explore the levels of player knowledge that are essential for playing the game. To facilitate our investigations on these knowledge levels, we discuss two kinds of strategies for the players, history-based and positional, and show what matters in this context. Our study eventually sheds light on the implicit assumptions prevalent in the existing literature in terms of player knowledge while playing the game, and provides a foundation for further studies on epistemically informed game structures.

1 Introduction

The cops and robber game is a well-known model for search missions and pursuit-evasion environments. It is independently introduced by Nowakowski and Winkler (1983) and Quillot (1978), and has become an important testbed for studying algorithms and computational complexity, with deep roots in computer science. The game is played on a graph by a robber and one or more cops. In each round, the players move along the edges of the graph. The goal of the cops is to move to the same node as the robber, while the robber aims to avoid the cops. So far, many variants of the game have been developed, including both perfect and imperfect information versions of the game. See (Bonato and Nowakowski, 2011) for an extensive survey.

In this work, we will focus on the imperfect information version of the game. More specifically, the cops and the robber in our game may not know each other's positions as they move on the game graph, with the cops trying to figure out where the robber is. Moreover, we assume that the players have observational ability as follows: players can see each other (so they know each other's positions) if they are at the same position or their positions are connected by an edge of the graph (that is, their distance is no more than 1). This is called 1-*sight ability* (also known as 1-*visibility*), and is proposed by Tang (2004). Limited sight abilities have also been explored in (Tošić, 1985; Clarke et al., 2020) considering players having k-sight ability with $k \geq 0$.

In contrast to the aforementioned articles that focus on the algorithmic and computational features of the variants, our purpose is to examine some fundamental principles of the game structure from both epistemic and game-theoretic perspectives. We are motivated by the existing variants of the cops and robber game (Bonato and Nowakowski, 2011). To the best of our knowledge, there is no adequate explanation for why they are designed in such a way rather than following other natural alternatives. For instance, there are different approaches to the notion of strategy in the field of game theory, including history-based strategies and positional strategies,[1] the existing literature only considers positional strategies of cops and robber. Another common feature is that the notion of knowledge, when discussed, is restricted to first-order knowledge (that is, the knowledge about each other's positions), and higher-order knowledge (that is, the knowledge about each other's knowledge) of players is ignored.[2] In what follows, we analyze these choices in the game structures and provide justifications for the same. The term 'factual knowledge' basically refers to first-order knowledge mentioned above, constituting players' knowledge at the current stage of the game.

To facilitate our understanding of these aspects, let us first provide an intuitive explanation of the game that is played on a finite graph in which every node can reach at least one node along an edge. We are not considering any dead-ends just for the sake of brevity. We also assume that there are two cops x_1, x_2 (females) and one robber y (male) - the analysis and results presented here can easily be generalized to a setting with multiple cops. The players are located at some nodes on the graph. In each round, cops move first simultaneously along the edges and then the robber moves. We

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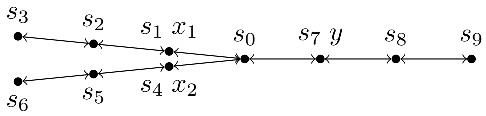
[1] Intuitively, a strategy is a function that tells a player how to act whenever its her turn to act. A history-based strategy is defined on the class of histories of play of a game, while a positional strategy only depends on the current position of the player in the game. We will provide formal definitions at a later stage.

[2] The two aspects involving strategy and knowledge are also common features of perfect-information variants. Regarding knowledge, they typically assume that players know each other's positions; however, there are no explicit discussions of the higher-order knowledge (Nowakowski and Winkler, 1983; Bonato and Nowakowski, 2011). Our study makes sense for those perfect information games as well.

assume that a player $z \in \{x_1, x_2, y\}$ always knows his/her position and has 1-*sight observational power*. The two cops have an understanding with each other, in that they fully exchange their knowledge about their own positions and the position of the robber: for instance, if x_1 knows that y is at s_1 or s_2, and x_2 knows that y is at s_2 or s_3, then they know that y is at s_2 (after communication). The knowledge shared by the two cops is called *distributed knowledge*, we will discuss the notion at a later stage. However, we note that the two cops might not know each other's positions before they exchange knowledge. To distinguish between the individual knowledge and the distributed knowledge of the cops, we will called the latter the *team knowledge of cops* (or simply *team knowledge*). We say that the *cops team knows that* φ when φ is their team knowledge. The goal of cops is to (distributively) know where the robber is - equivalently, to convert the robber's position into their team knowledge - while the robber tries to avoid that. Here is an example.

Example 1. *In the graph below, players* x_1, x_2 *and* y *are at* s_1, s_4 *and* s_7, *respectively. In what follows, we will use triples of nodes for the positions of* x_1, x_2 *and* y *(in this order), and so now we have* (s_1, s_4, s_7).



- *We first consider the knowledge of the two cops:*

  Cop x_1 *knows that she is at* s_1. *Based on the* 1-*sight ability, she can see that* x_2 *and* y *are not at* s_0, s_1 *or* s_2, *and thinks all other nodes are possible to be their positions.*

  Similarly, x_2 *knows that her own position is* s_4. *Moreover, she knows that* x_1 *and* y *are not at* s_0, s_4 *or* s_5, *and thinks all other nodes are possible to be their positions.*

  Therefore, after exchanging their knowledge with each other, the cops team knows that their own positions are s_1 *and* s_4 *and that* y *is at some node of* $\{s_3, s_6, s_7, s_8, s_9\}$.

- *Next, the knowledge of the robber* y *is as follows:*

  He knows where he is, i.e., s_7. *Based on the observational power, he knows that neither* x_1 *nor* x_2 *is at* s_0, s_7 *or* s_8, *but cannot rule out other possibilities.*

Now let the game begin.

- *Assume that the cops move to the left simultaneously. Based on the* 1-*sight ability, the movement makes them know more: the cops team knows that* y *is not at* s_3 *or* s_6, *and that* y *is at some node of* $\{s_7, s_8, s_9\}$.

  Now y *knows that neither* x_1 *nor* x_2 *is at* s_0, s_7, s_8 *or* s_9*: he can see* s_0, s_7 *and* s_8 *directly; and* s_9 *can only be reached from* s_8, *but before the movement,* y *knew that cops were not there, so after the movement,* y *knows that neither* x_1 *nor* x_2 *is at* s_9. *Thus* y *knows that the cops are at some nodes of* $\{s_1, s_2, s_3, s_4, s_5, s_6\}$.
- *Next, let* y *move to* s_8. *After the first round, their positions are* (s_2, s_5, s_8).
- *Assume that in the subsequent stages the robber always moves between* s_8 *and* s_9. *The cops then have a strategy ensuring their win: they just need to move to the right, and at some stage they would get to know where* y *is.*

Outline of the paper. Section 2 presents the formal details for the game, and explores its basic properties. Section 3 develops two notions of strategies, history-based and positional, and proves that they are equivalent with respect to the existence of winning strategies. In Section 4, we introduce the game with higher-order knowledge. Section 5 explores the corresponding notions of history-based strategy and the positional strategy, shows their equivalence, and proves that the second-order knowledge does not matter. Finally, we conclude with some further directions in Section 6.

2 Preliminaries

Let us first focus on a formal presentation of the cops and robber game that is explored in the article. The designs uses the notion of *distributed knowledge* (of the cops) as a primitive, which intuitively means the knowledge of the players that is obtained after sharing their individual knowledge with each other. This usage of the notion is different from the ordinary approach in the literature (van Benthem, 2011; van Ditmarsch, van der Hoek, and Kooi, 2008; Fagin et al., 1995), which uses the knowledge of individual players as a primitive and regards distributed knowledge as a derived notion. One can also develop a design for the game with individual knowledge as a primitive notion and show that the two representations are equivalent, which would be useful in indicating that the notion used by us does capture the distributed knowledge in the standard sense. However, compared with the individual knowledge of cops, only their distributed knowledge is crucial for the game; therefore, our design that represents only distributed knowledge is a more direct and succinct representation of the key information.

As stated in Section 1, we consider the setting with two cops x_1, x_2 (females) and a robber y (male), and the analysis and results developed can easily be generalized to the case with multiple cops. We write Var for the set $\{x_1, x_2, y\}$ of players, C for $\{x_1, x_2\}$ and R for $\{y\}$.

The game is played on a graph, called a *game board*, defined in the following manner:

Definition 2. *A* game board *is a tuple* $(\mathbf{D}, \mathbf{R})$, *where* $\mathbf{D}$ *is a non-empty finite set of nodes (also called positions or states) and* $\mathbf{R} \subseteq \mathbf{D} \times \mathbf{D}$ *is a serial binary relation, that is, for any* $s \in \mathbf{D}$, *there is* $t \in \mathbf{D}$ *with* $(s, t) \in \mathbf{R}$.

As stated, players are assumed to have the 1-*sight observational power*: a player z can see the players who are at the same position as z or at positions that can be reached in one step via $\mathbf{R}$ or the converse of $\mathbf{R}$. For any $s \in \mathbf{D}$, we define

$$\mathbb{S}(s) := \{s\} \cup \{t \in \mathbf{D} \mid (s, t) \in \mathbf{R} \text{ or } (t, s) \in \mathbf{R}\},$$

describing the observable region of a player located at s. We call $\sigma \in \mathbf{D}^{\mathsf{Var}}$ a *situation* (or *assignment*) of the game, which characterizes the positions of the players: for instance, when $\sigma(x_1) = s_1$, $\sigma(x_2) = s_4$ and $\sigma(y) = s_7$, σ describes the initial situation in Example 1. As the case in the example, we often use triples of nodes (e.g., (s_1, s_4, s_7)) for assignments.

Definition 3. *A* cops and robber game *is a tuple* $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi)$, *where* $(\mathbf{D}, \mathbf{R})$ *is a game board, and* $\Pi =$

$\{\Sigma_\mathsf{C}, \Sigma_\mathsf{R}\}$ *consists of the* information states $\Sigma_\mathsf{C} \subseteq \mathbf{D}^{\mathsf{Var}}$ *of the cops team and* $\Sigma_\mathsf{R} \subseteq \mathbf{D}^{\mathsf{Var}}$ *of the robber such that*

- (`G1`) *For any* $\sigma_1, \sigma_2 \in \Sigma_\mathsf{R}$ *and any* $z \in \mathsf{Var}$*, if* $\sigma_1(z) \in \mathbb{S}(\sigma_1(y))$*, then* $\sigma_2(z) = \sigma_1(z)$*.*

  [Σ_R *consists of those* $\sigma \in \mathbf{D}^{\mathsf{Var}}$ *that are considered to be possible by the robber* y*. When* z *is* y*,* $\sigma(z) \in \mathbb{S}(\sigma(y))$ *holds for any* σ*, and then the clause ensures that* y *has the same position in all situations of* Σ_R*, meaning that* y *always knows where* y *is. When* z *is a cop, if* z *is in the sight of* y *in some* $\sigma_1 \in \Sigma_\mathsf{R}$*, then* z *has the same position in all possible situations: intuitively, every player knows whether another player is in the sight (so is* y*), and once* y *considers such a* σ_1 *to be possible,* y *knows where* z *is.]*
- (`G2`) *For any* $\sigma_1, \sigma_2 \in \Sigma_\mathsf{C}$ *and any* $x \in \{x_1, x_2\}$*, it holds that* $\sigma_1(x) = \sigma_2(x)$ *and that if* $\sigma_1(y) \in \mathbb{S}(\sigma_1(x))$ *or* $\sigma_2(y) \in \mathbb{S}(\sigma_2(x))$*, then* $\sigma_1 = \sigma_2$*.*

  [Σ_C *represents the team knowledge (i.e., distributed knowledge) of the cops. The first part ensures that the cops team knows the positions of the cops. The only uncertainty of the cops team is about the position of* y*, and as described by the second part, once a cop can see* y*, the cops team knows the actual situation.]*
- (`G3`) $\Sigma_\mathsf{C} \cap \Sigma_\mathsf{R} \neq \emptyset$*.*

  [There is at least one situation that is considered to be possible by both the cops team and the robber.]

What the information states show is the *first-order knowledge* of the players (i.e., their knowledge about each other's positions), but not *higher-order knowledge* (i.e., the knowledge about each other's knowledge). At a later stage, we will discuss the latter notion. For now, we consider an example.

Example 4. *We present the initial information states* Σ^1_C *and* Σ^1_R *for the game described in Example 1.*

According to the previous analysis, the information state Σ^1_C *of cops team is* $\{\sigma_3, \sigma_6, \sigma_7, \sigma_8, \sigma_9\}$*, where for any* $i \in \{3, 6, 7, 8, 9\}$*,* $\sigma_i(x_1) = s_1$*,* $\sigma_i(x_2) = s_4$ *and* $\sigma_i(y) = s_i$*. Any cop has the same positions in different situations of* Σ^1_C*, and this means that the positions of the cops are their team knowledge: although they are not in the sight of each other, knowing each other's positions can be seen as an outcome of their exchange of information. Another outcome is that the cops team considers a node to be a possible position of the robber* y *only if the node is not observed by either of them.*

Moreover, neither x_1 *nor* x_2 *is in the observational region* $\{s_0, s_7, s_8\}$ *of* y *from* s_7*, and from his perspective, the rest of the nodes are possible positions of the cops. Also,* y *knows where he is. Thus the robber's knowledge can be characterized by the information state* $\Sigma^1_\mathsf{R} := \{\sigma \in \mathbf{D}^{\mathsf{Var}} \mid \sigma(y) = s_7$ *and* $\sigma(x) \notin \{s_0, s_7, s_8\}$ *for any* $x \in \mathsf{C}\}$*.*

At first glance, the definition above does not give information about the actual positions of the players, but in effect, they are provided in an implicit but precise way. To make this clear, let us consider (`G3`), and we have the following:

Proposition 5. *In a cops and robber game, the following statements are equivalent:*

- (i) $\Sigma_\mathsf{C} \cap \Sigma_\mathsf{R} \neq \emptyset$
- (ii) $\Sigma_\mathsf{C} \cap \Sigma_\mathsf{R}$ *is a singleton set.*

Proof. The direction from (ii) to (i) is trivial. For the converse, suppose that there are distinct $\sigma, \sigma' \in \Sigma_\mathsf{C} \cap \Sigma_\mathsf{R}$. Then, there exists at least one variable $z \in \mathsf{Var}$ such that $\sigma(z) \neq \sigma'(z)$. However, when z is y, it contradicts the condition (`G1`), and when $z \in \{x_1, x_2\}$, it contradicts (`G2`). Thus, $\Sigma_\mathsf{C} \cap \Sigma_\mathsf{R}$ is a singleton set, as needed. □

Given a game $(\mathbf{D}, \mathbf{R}, \Pi)$, the unique element of $\bigcap \Pi$ (i.e., $\Sigma_\mathsf{C} \cap \Sigma_\mathsf{R}$) would give us exact information about the actual positions of the players, which represents the actual situation of the game. Clause (`G3`) ensures that no player ignores the actual situation. To highlight the actual positions, we would often write $(\mathbf{D}, \mathbf{R}, \Pi, \sigma)$, where $\sigma \in \bigcap \Pi$.

Let us now describe how a game runs. For this, we need to define the actions of the players. Movements of players may change not only the positions of the players who move, but also the information states. To describe the changes of positions, let us consider the relations M^V among situations: for any non-empty $V \subseteq \mathsf{Var}$ and any $\sigma, \sigma' \in \mathbf{D}^{\mathsf{Var}}$,

$$\mathsf{M}^V \sigma\sigma' \quad \text{iff} \quad \begin{array}{l} \text{for any } z \in V,\ (\sigma(z), \sigma'(z)) \in \mathbf{R}, \text{ and} \\ \text{for any } z' \notin V,\ \sigma(z) = \sigma'(z). \end{array}$$

So, when $\mathsf{M}^V \sigma\sigma'$, the positions of $z \in V$ in situation σ' are $\mathbf{R}$-successors of the positions of the same in σ, while the positions of $z \notin V$ remain unchanged. This describes the changes in positions after simultaneous movement of members of V.[3] To capture how the information states are affected by movements, given a set $\Sigma \subseteq \mathbf{D}^{\mathsf{Var}}$ of situations and a set V of variables, we define the following notion:

$$\mathsf{M}^V(\Sigma) := \{\sigma' \in \mathbf{D}^{\mathsf{Var}} \mid \textit{there is a } \sigma \in \Sigma \textit{ s.t. } \mathsf{M}^V \sigma\sigma'\}.$$

For brevity, when Σ is a singleton set $\{\sigma\}$, we write $\mathsf{M}^V(\sigma)$. For any Σ, $\mathsf{M}^V(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathsf{M}^V(\sigma)$.

Now we introduce the notion of actions of the players:

Definition 6. *Let* $\mathcal{G}_1 = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ *be a game and* $\star \in \{\mathsf{C}, \mathsf{R}\}$*. An* action *(or* movement*) of* $\star$ *from* $\mathcal{G}_1$ *updates the game into a new game* $\mathcal{G}_2 = (\mathbf{D}, \mathbf{R}, \Pi_2, \sigma_2)$*, where*

- (`Act1`) $\sigma_2 \in \mathsf{M}^\star(\sigma_1)$

  [Player(s) $\star$ *can only move along* $\mathbf{R}$*, so the new actual situation must be an* $\mathsf{M}^\star$*-successor of the previous* σ_1*.]*
- (`Act2`) *The new information state* $\Sigma'_\mathsf{R} \in \Pi_2$ *of the robber consists of those* $\sigma \in \mathsf{M}^\star(\Sigma_\mathsf{R})$ *such that for any* $z \in \mathsf{Var}$*:*
  - *If* $\sigma_2(z) \in \mathbb{S}(\sigma_2(y))$*, then* $\sigma(z) = \sigma_2(z)$*.*
  - *If* $\sigma_2(z) \notin \mathbb{S}(\sigma_2(y))$*, then* $\sigma(z) \notin \mathbb{S}(\sigma(y))$*.*

  [Given the new actual situation, whether a player z *is in the sight of* y *is determined. If* y *can see* z *directly, then* y *knows the position of* z*; otherwise* y *considers a situation possible only if* z *in that situation is not in the sight of* y*.]*
- (`Act3`) *The new information state* $\Sigma'_\mathsf{C} \in \Pi_2$ *of the cops team consists of the situations* $\sigma \in \mathsf{M}^\star(\Sigma_\mathsf{C})$ *such that*
  - *For any* $x \in \mathsf{C}$*,* $\sigma(x) = \sigma_2(x)$*.*
  - *If* $\sigma_2(y) \in \mathbb{S}(\sigma_2(x_1)) \cup \mathbb{S}(\sigma_2(x_2))$*, then* $\sigma = \sigma_2$*.*
  - *If* $\sigma_2(y) \notin \mathbb{S}(\sigma_2(x_1)) \cup \mathbb{S}(\sigma_2(x_2))$*, then it holds that* $\sigma(y) \notin \mathbb{S}(\sigma(x_1)) \cup \mathbb{S}(\sigma(x_2))$*.*

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[3] When V is singleton set $\{z\}$, it means that after the movement of z, σ becomes σ'.

[The new Σ'_{C} still aims to capture the distributed knowledge of the cops. Since every player always knows her/his own position, the positions of the cops given by the situations in Σ'_{C} are always the same as those given by the new actual situation σ_2. Moreover, similar to the case of Σ'_{R}, given σ_2, whether y is in the sight of a cop is determined.]

What the new information states are depends on the new actual situation after the action (as well as the information states before the action). For example, when it is the turn of cops, what they need to do is just move to the successors of their own positions, and then the new information states will be formed automatically according to (`Act2`) and (`Act3`); we note that movements to different positions may lead to different information states for all players.

Example 7. *Recall that Example 4 has identified the initial information states Σ^1_{C} and Σ^1_{R} of the game given by Example 1. We show how the information states are updated after x_1 and x_2 move from s_1 and s_4 to s_2 and s_5, respectively.*

We use σ_2 for the new situation after the movements, i.e., $\sigma_2 = (s_2, s_5, s_7)$. According to the analysis in Example 1, the new information state Σ^2_{C} of the cops team consists of

$$\sigma_2 = (s_2, s_5, s_7),\ \sigma_a = (s_2, s_5, s_8) \text{ and } \sigma_b = (s_2, s_5, s_9),$$

and the new information state Σ^2_{R} of the robber is as follows:

$$\{\sigma_{ij} \mid 1 \le i, j \le 6, \sigma_{ij}(x_1) = s_i, \sigma_{ij}(x_2) = s_j, \sigma_{ij}(y) = s_7\}.$$

For any $\star \in \{\mathsf{C}, \mathsf{R}\}$, $\sigma_2 \in \Sigma^2_\star \subseteq \mathsf{M}^{\mathsf{C}}(\Sigma^1_\star)$. Σ^2_{R} and Σ^2_{C} can be obtained by the calculation described by (`Act2`) *and* (`Act3`) *respectively, which picks out from $\mathsf{M}^{\mathsf{C}}(\Sigma^1_\star)$ the situations that are 'consistent with' the sight of $\star$ from the new position in σ_2. For instance, although $(s_2, s_5, s_3) \in \mathsf{M}^{\mathsf{C}}(\Sigma^1_{\mathsf{C}})$, Σ^2_{C} does not contain it, since the situation is such a case that a cop can see the robber while σ_2 is not such a case.*

Remark 8. *Let us spell out the underlying assumptions implicitly taken by the definition for actions.*

- *According to Definition 6, when a player z moves, not only is the information state of z updated, but also are those of other players. This indicates that players know that whose turn it is to act,[4] although other players may not know the origin and the destination of the moves.*
- *As described by* (`Act2`) *and* (`Act3`)*, when z moves, all other players update their own information states, by replacing the positions of z considered to be possible by them before the movement with the successors of those positions. To achieve this, all players should know the structure of the game board.*
- *After an action, the new information states are determined by the new actual situation and the information states before the action. The latter requires that they cannot forget what are considered to be possible before the action.*

It is important to point out that the $\mathcal{G}_2$ above, resulting from the implementation of an action on $\mathcal{G}_1$, is well defined:

Proposition 9. *Let $\mathcal{G}_1 = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ be a cops and robber game, $\star \in \{\mathsf{C}, \mathsf{R}\}$ and $\sigma_2 \in \mathsf{M}^\star(\sigma_1)$. Then, $\mathcal{G}_2 = (\mathbf{D}, \mathbf{R}, \Pi_2 = \{\Sigma^2_{\mathsf{C}}, \Sigma^2_{\mathsf{R}}\}, \sigma_2)$ is well defined, where Π_2 is given by* (`Act2`) *and* (`Act3`)*. Moreover, $\{\sigma_2\} = \bigcap \Pi_2$.*

Proof. (1) We start by showing that $\mathcal{G}_2$ is well defined. Let us first consider (`G3`). By Definition 6, given a fixed situation $\sigma_2 \in \mathsf{M}^\star(\sigma_1)$, for the associated Π_2, it is always the case that $\sigma_2 \in \bigcap \Pi_2$. So the condition is satisfied.

Next, we move to (`G1`). Let $\sigma, \sigma' \in \Sigma^2_{\mathsf{R}}$ and $z \in \mathsf{Var}$ s.t. $\sigma(z) \in \mathbb{S}(\sigma(y))$. For the new actual situation σ_2, according to (`Act2`), we have $\sigma_2(z) \in \mathbb{S}(\sigma_2(y))$ and $\sigma(z) = \sigma_2(z)$. Using again the clause, we get $\sigma'(z) = \sigma_2(z) = \sigma(z)$.

Finally, we prove that (`G2`) is satisfied. Let $\sigma, \sigma' \in \Sigma^2_{\mathsf{C}}$ and $x \in \mathsf{C}$. By the first sub-clause of (`Act3`), $\sigma(x) = \sigma'(x)$. Moreover, let $\sigma(y) \in \mathbb{S}(\sigma(x))$. Then, by the second sub-clause of the condition, it holds that $\sigma = \sigma' = \sigma_2$.

(2) As stated, $\sigma_2 \in \bigcap \Pi_2$, so $\bigcap \Pi_2 \neq \emptyset$. Since both (`G1`) and (`G2`) are satisfied, it follows from Proposition 5 that $\{\sigma_2\} = \bigcap \Pi_2$, as desired. $\square$

3 Strategies

We now proceed to define the notion of strategies for the game. There are two widely used approaches to the notion: on the basis of (i) the full history of the game play (called *history-based strategy*), and (ii) the current stages of the game play (called *positional strategy*). We develop the corresponding notions for the cops and robber game, and prove that they are equivalent in this case regarding the existence of winning strategies. Let us begin with the following:

Definition 10. *Given a game $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$, a* run *of the game is an infinite sequence of pairs (σ, Σ) consisting of a situation and a set of information states of the players:*

$$(\sigma_1, \Pi_1), (\sigma_2, \Pi_2), \ldots, (\sigma_m, \Pi_m), \ldots$$

such that

(`Run1`) *For each odd number $i \ge 1$ such that $\Sigma^i_{\mathsf{C}} \in \Pi_i$ is not a singleton set, $(\mathbf{D}, \mathbf{R}, \Pi_{i+1}, \sigma_{i+1})$ is a game updated from $(\mathbf{D}, \mathbf{R}, \Pi_i, \sigma_i)$ by an action of the cops.*

(`Run2`) *For each even number $i \ge 1$ such that $\Sigma^i_{\mathsf{C}} \in \Pi_i$ is not a singleton set, $(\mathbf{D}, \mathbf{R}, \Pi_{i+1}, \sigma_{i+1})$ is a game updated from $(\mathbf{D}, \mathbf{R}, \Pi_i, \sigma_i)$ by an action of the robber.*

(`Run3`) *When $\Sigma^i_{\mathsf{C}} \in \Pi_i$ is a singleton set (i.e., $\{\sigma_i\}$) for some $i \in \mathbb{N}$, $(\sigma_j, \Pi_j) = (\sigma_i, \Pi_i)$ for all $j > i$.*

Given a run, cops win *if $\Sigma^i_{\mathsf{C}} = \{\sigma_i\}$ for some $i \in \mathbb{N}$; otherwise,* robber wins.

Conditions (`Run1`) and (`Run2`) indicate that (before the cops team wins) the cops and the robber take turns to act and that cops act first. (`Run3`) means that when the cops team knows robber's position, no further actions can be taken.

Given a game, for any $\star \in \{\mathsf{C}, \mathsf{R}\}$, by saying that *the information state $\Sigma_\star$ is determined by the sight of $\star$*, we mean that $\Sigma_\star$ is given by the 1-sight observational power of $\star$ from the position(s) in the actual situation. For instance, when the robber is in the sight of a cop, the information state of the cops team is a singleton set given by the actual situation, and when he is not in the sight of any cop, the team considers all unobservable nodes to be a possible position for robber. Moreover, by saying *an initial game*, we mean that Σ_{C} and Σ_{R} are determined by the sight of C and R, respectively.

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[4] This is a common assumption of ordinary turn-based games, which is often referred to as *synchronicity* (see e.g., Halpern, 2007).

Given an initial game $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi = \{\Sigma_{\mathsf{C}}, \Sigma_{\mathsf{R}}\}, \sigma)$, players' uncertainty indicates that they may not know which game they will play: for instance, given a $\sigma' \in \Sigma_{\mathsf{C}}$ that is distinct from σ, the cops team thinks it is possible that the game they are playing is $\mathcal{G}' = (\mathbf{D}, \mathbf{R}, \{\Sigma_{\mathsf{C}}, \Sigma_{\mathsf{R}}^{\sigma'}\}, \sigma')$, where $\Sigma_{\mathsf{R}}^{\sigma'}$ is determined by the sight of the robber from $\sigma'(y)$.

Definition 11. *Let $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi = \{\Sigma_{\mathsf{C}}, \Sigma_{\mathsf{R}}\}, \sigma)$ be an initial game.*

We use $\mathbb{G}_{\mathsf{C}}^{\mathcal{G}}$ for the games considered to be possible by the cops team, i.e., $\{\mathcal{G}' = (\mathbf{D}, \mathbf{R}, \Pi' = \{\Sigma_{\mathsf{C}}, \Sigma_{\mathsf{R}}^{\sigma'}\}, \sigma') \mid \sigma' \in \Sigma_{\mathsf{C}},$ and $\Sigma_{\mathsf{R}}^{\sigma'}$ is determined by the sight of y from $\sigma'(y)\}$.

We use $\mathbb{G}_{\mathsf{R}}^{\mathcal{G}}$ for the games considered to be possible by the robber, i.e., $\{\mathcal{G}' = (\mathbf{D}, \mathbf{R}, \Pi' = \{\Sigma_{\mathsf{C}}^{\sigma'}, \Sigma_{\mathsf{R}}\}, \sigma') \mid \sigma' \in \Sigma_{\mathsf{R}},$ and $\Sigma_{\mathsf{C}}^{\sigma'}$ is determined by the sight of cops from $\sigma'(x_1)$ and $\sigma'(x_2)\}$.

One can check that all games in $\mathbb{G}_{\mathsf{C}}^{\mathcal{G}} \cup \mathbb{G}_{\mathsf{R}}^{\mathcal{G}}$ are well defined.

3.1 Positional strategies

We start by discussing the positional strategies for the game, which only depend on the current situations and information states of the players. Formally, the details are as follows:

Definition 12. *Let $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ be a game, and let*

$$\mathbb{P} = \{(\sigma, \Pi) \mid \sigma \in \bigcap \Pi, \text{ and } \Pi \text{ satisfies } (\mathtt{G1}) \text{ and } (\mathtt{G2})\}.$$

A positional strategy for the cops team *is a function $f : \mathbb{P} \to \mathbb{P}$ such that*

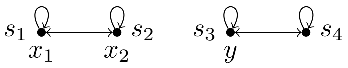
- $(\mathtt{PS1})$ *$f(\sigma, \Pi) = (\sigma', \Pi')$ only if $\Sigma_{\mathsf{C}} \in \Pi$ is not a singleton set, $\sigma' \in \mathsf{M}^{\mathsf{C}}(\sigma)$, and Π' is the class of information states from the game $(\mathbf{D}, \mathbf{R}, \Pi', \sigma')$ obtained by updating $(\mathbf{D}, \mathbf{R}, \Pi, \sigma)$ with the movement of cops from $\sigma(x_1)$ and $\sigma(x_2)$ to, respectively, $\sigma'(x_1)$ and $\sigma'(x_2)$.*
- $(\mathtt{PS2})$ *For any pairs $(\sigma_a, \Pi_a), (\sigma_b, \Pi_b), (\sigma_c, \Pi_c), (\sigma_d, \Pi_d)$ from $\mathbb{P}$, when $\Sigma_{\mathsf{C}}^a \in \Pi_a$ and $\Sigma_{\mathsf{C}}^c \in \Pi_c$ are the same, $f(\sigma_a, \Pi_a) = (\sigma_b, \Pi_b)$ and $f(\sigma_c, \Pi_c) = (\sigma_d, \Pi_d)$, it holds that $\sigma_b(x) = \sigma_d(x)$ for any $x \in \mathsf{C}$.*

Similarly, a positional strategy for the robber *is also a function $g : \mathbb{P} \to \mathbb{P}$ such that*

- $(\mathtt{PS3})$ *$g(\sigma, \Pi) = (\sigma', \Pi')$ only if $\Sigma_{\mathsf{C}} \in \Pi$ is not a singleton set, $\sigma' \in \mathsf{M}^{\mathsf{R}}(\sigma)$, and Π' is the class of information states from the game $(\mathbf{D}, \mathbf{R}, \Pi', \sigma')$ obtained by updating $(\mathbf{D}, \mathbf{R}, \Pi, \sigma)$ with the movement of y from $\sigma(y)$ to $\sigma'(y)$.*
- $(\mathtt{PS4})$ *For any pairs $(\sigma_a, \Pi_a), (\sigma_b, \Pi_b), (\sigma_c, \Pi_c), (\sigma_d, \Pi_d)$ from $\mathbb{P}$, when $\Sigma_{\mathsf{R}}^a \in \Pi_a$ and $\Sigma_{\mathsf{R}}^c \in \Pi_c$ are the same, $g(\sigma_a, \Pi_a) = (\sigma_b, \Pi_b)$ and $g(\sigma_c, \Pi_c) = (\sigma_d, \Pi_d)$, it holds that $\sigma_b(y) = \sigma_d(y)$.*

Informally, a positional strategy for the cops team is a function telling them where they move to. Similarly for a positional strategy for the robber. Clauses $(\mathtt{PS1})$ and $(\mathtt{PS3})$ indicate that actions can be taken only when cops have not won yet, and ensure that actions taken according to a strategy are well defined. By $(\mathtt{PS2})$ and $(\mathtt{PS4})$, if the positions and information states of the players in (σ_a, Π_a) and (σ_b, Π_b) are the same, then starting from (σ_a, Π_a) and (σ_b, Π_b) they can move only to the same positions. This illustrates that strategies depend on the information available to the players.

Example 13. *The movement patterns described in Example 1 can be formally represented as positional strategies for the players. Let us consider another example as depicted below:*



We write σ_1 for this initial situation (s_1, s_2, s_3). Also, the initial information states $\Pi_1 = \{\Sigma_{\mathsf{C}}^1, \Sigma_{\mathsf{R}}^1\}$ are as follows:

$$\Sigma_{\mathsf{C}}^1 = \{(s_1, s_2, s_3), (s_1, s_2, s_4)\},$$
$$\Sigma_{\mathsf{R}}^1 = \{(s_1, s_1, s_3), (s_1, s_2, s_3), (s_2, s_1, s_3), (s_2, s_2, s_3)\}.$$

Assume that cops always travel between s_1 and s_2. After the movement of cops, the new situation is $\sigma_2 = (s_2, s_1, s_3)$. The new information states are

$$\Sigma_{\mathsf{C}}^2 = \{(s_2, s_1, s_3), (s_2, s_1, s_4)\} \quad \text{and} \quad \Sigma_{\mathsf{R}}^2 = \Sigma_{\mathsf{R}}^1.$$

Suppose that y acts based on a strategy g, and he stays at node s_3. After this, the situation is $\sigma_3 = (s_2, s_1, s_3)$, and

$$\Sigma_{\mathsf{C}}^3 = \Sigma_{\mathsf{C}}^1 \quad \text{and} \quad \Sigma_{\mathsf{R}}^3 = \Sigma_{\mathsf{R}}^1.$$

Then, cops still act according to the strategy. After their movement, the actual situation is $\sigma_4 = \sigma_1$, and the information states are $\Sigma_{\mathsf{C}}^4 = \Sigma_{\mathsf{C}}^1$ and $\Sigma_{\mathsf{R}}^4 = \Sigma_{\mathsf{R}}^1$. Now, is it possible that y moves to s_4 according to g? The answer is negative: although $(\sigma_2, \Pi_2) \neq (\sigma_4, \Pi_4)$, since $\sigma_2(y) = \sigma_4(y)$ and $\Sigma_{\mathsf{R}}^2 = \Sigma_{\mathsf{R}}^4$, by $(\mathtt{PS4})$ player y can only stay at s_3.

Let us introduce the corresponding notion of positional winning strategy, which is a strategy that can ensure the winning of the players regardless of their uncertainty:

Definition 14. *Let $\mathcal{G}$ be an initial game. A* positional winning strategy for the cops team *is a strategy s.t. when they follow it, they can win all runs of the games in $\mathbb{G}_{\mathsf{C}}^{\mathcal{G}}$. A* positional winning strategy for the robber *is a strategy s.t. when he follows it, he can win all runs of the games in $\mathbb{G}_{\mathsf{R}}^{\mathcal{G}}$.*

For illustration, the strategy for the cops team described in Example 1 can be seen as a positional winning strategy for the team, and any positional strategy for the robber in Example 13 is a winning strategy for him.

3.2 History-based strategies

Next, we explore history-based strategies. Intuitively, a *history* records what has happened in the game play, and it is a finite initial fragment of a run such that the cops team does not win or just wins at the last pair (σ, Σ) of the sequence.

Given a game $\mathcal{G}$, we use $\mathcal{H}_{\mathsf{C}}^{\mathcal{G}}$ for the class of histories whose first pair is given by some game from $\mathbb{G}_{\mathsf{C}}^{\mathcal{G}}$, $(\mathcal{H}_{\mathsf{C}}^{\mathcal{G}})^{\mathsf{odd}}$ for the histories of $\mathcal{H}_{\mathsf{C}}^{\mathcal{G}}$ with odd lengths (so cops need to act), and $(\mathcal{H}_{\mathsf{C}}^{\mathcal{G}})^{\mathsf{even}}$ for those of $\mathcal{H}_{\mathsf{C}}^{\mathcal{G}}$ with even lengths (so the robber needs to act). Similar, we use $\mathcal{H}_{\mathsf{R}}^{\mathcal{G}}$ for the class of histories whose first pair is given by some game from $\mathbb{G}_{\mathsf{R}}^{\mathcal{G}}$, and use $(\mathcal{H}_{\mathsf{R}}^{\mathcal{G}})^{\mathsf{odd}}$ and $(\mathcal{H}_{\mathsf{R}}^{\mathcal{G}})^{\mathsf{even}}$ for the histories of $\mathcal{H}_{\mathsf{R}}^{\mathcal{G}}$ with, respectively, odd lengths and even lengths.

Definition 15. *Let $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ be an initial game.*

A history-based strategy for the cops team *is a function $f : (\mathcal{H}_{\mathsf{C}}^{\mathcal{G}})^{\mathsf{odd}} \to (\mathcal{H}_{\mathsf{C}}^{\mathcal{G}})^{\mathsf{even}}$ such that the image of*

$$(\sigma'_1,\Pi'_1),\ldots,(\sigma'_{2n},\Pi'_{2n}),(\sigma'_{2n+1},\Pi'_{2n+1})$$

is obtained by adding a pair $(\sigma'_{2n+2},\Pi'_{2n+2})$ *to the end of the history, where*

- (`HS1`) *For any* $1 \leq i \leq 2n+1$, *there are* $\sigma,\sigma' \in \Sigma^i_\mathsf{C} \in \Pi'_i$ *with* $\sigma(y) \neq \sigma'(y)$.
- (`HS2`) $\sigma'_{2n+2} \in \mathsf{M}^\mathsf{C}(\sigma'_{2n+1})$ *and* Π'_{2n+2} *is the resulting set of information states.*
- (`HS3`) *For any*
  $h_a = (\sigma^a_1,\Pi^a_1),\ldots,(\sigma^a_{2n+1},\Pi^a_{2n+1})$
  $h_b = h_a,(\sigma^a_{2n+2},\Pi^a_{2n+2})$
  $h_c = (\sigma^c_1,\Pi^c_1),\ldots,(\sigma^c_{2n+1},\Pi^c_{2n+1})$
  $h_d = h_c,(\sigma^c_{2n+2},\Pi^c_{2n+2})$
  such that $f(h_a) = h_b$ *and* $f(h_c) = h_d$, *if for any* $1 \leq i \leq 2n+1$, $\Sigma^a_\mathsf{C} \in \Pi^a_i$ *is the same as* $\Sigma^c_\mathsf{C} \in \Pi^c_i$, *then* $\sigma^a_{2n+2}(x) = \sigma^c_{2n+2}(x)$ *for any cop* $x \in \mathsf{C}$.

Similarly, a history-based strategy for the robber *is a map* $g : (\mathcal{H}^\mathcal{G}_\mathsf{R})^{\mathsf{even}} \to (\mathcal{H}^\mathcal{G}_\mathsf{R})^{\mathsf{odd}}$ *such that the image of*

$$(\sigma'_1,\Pi'_1),\ldots,(\sigma'_{2m},\Pi'_{2m})$$

is obtained by adding a pair $(\sigma'_{2m+1},\Pi'_{2m+1})$ *to the end of the history, where*

- (`HS4`) *For any* $1 \leq j \leq 2m$, *there are* $\sigma,\sigma' \in \Sigma^i_\mathsf{C} \in \Pi'_i$ *with* $\sigma(y) \neq \sigma'(y)$.
- (`HS5`) $\sigma'_{2m+1} \in \mathsf{M}^\mathsf{R}(\sigma'_{2m})$ *and* Π'_{2m+1} *is the resulting set of information states.*
- (`HS6`) *For any*
  $h_a = (\sigma^a_1,\Pi^a_1),\ldots,(\sigma^a_{2m},\Pi^a_{2m})$
  $h_b = h_a,(\sigma^a_{2m+1},\Pi^a_{2m+1})$
  $h_c = (\sigma^c_1,\Pi^c_1),\ldots,(\sigma^c_{2m},\Pi^c_{2m})$
  $h_d = h_c,(\sigma^c_{2m+1},\Pi^c_{2m+1})$
  with $g(h_a) = h_b$ *and* $g(h_c) = h_d$, *if for any* $1 \leq i \leq 2m$, $\Sigma^a_\mathsf{R} \in \Pi^a_i$ *is the same as* $\Sigma^c_\mathsf{R} \in \Pi^c_i$, *then it holds that* $\sigma^a_{2m+1}(y) = \sigma^c_{2m+1}(y)$.

By (`HS1`) and (`HS4`), these functions only make sense for the sequences containing no pairs representing the case that the cops team already wins. Moreover, (`HS3`) and (`HS6`) indicate that their actions depend on the information available to them. For instance, given two histories of the same length, if the cops team always has the same information states at the corresponding stages (intuitively this means that the cops team cannot distinguish between the histories), then by (`HS3`), where they will move to should be the same.[5]

Definition 16. *Let* $\mathcal{G} = (\mathbf{D},\mathbf{R},\Pi_1,\sigma_1)$ *be an initial game. A* history-based winning strategy for the cops team *is a strategy such that when they follow it, they can win all runs of the games in* $\mathbb{G}^\mathcal{G}_\mathsf{C}$. *A* history-based winning strategy for robber *is a strategy such that when he follows it, he can win all runs of the games in* $\mathbb{G}^\mathcal{G}_\mathsf{R}$.

Histories do not matter There is an equivalence between the two notions of strategies, in the sense of the following:

Theorem 17. *Let* $\mathcal{G} = (\mathbf{D},\mathbf{R},\Pi_1 = \{\Sigma^1_\mathsf{C},\Sigma^1_\mathsf{R}\},\sigma_1)$ *be an initial game. Then, it holds that*

- (i) *The cops team has a history-based winning strategy iff the team has a positional winning strategy.*
- (ii) *The robber has a history-based winning strategy iff he has a positional winning strategy.*

Proof. The proofs for the two parts are similar, and we merely consider the former.

(1) Assume that the team has a positional winning strategy g. Define a history-based strategy f of the team as follows: for any history $h_1 = (\sigma'_1,\Pi'_1),\ldots,(\sigma'_{2n+1},\Pi'_{2n+1})$ from $\mathcal{H}^\mathcal{G}_\mathsf{C}$, $f(h_1) = h_1,(\sigma'_{2n+2},\Pi'_{2n+2})$ if and only if $g(\sigma'_{2n+1},\Pi'_{2n+1}) = (\sigma'_{2n+2},\Pi'_{2n+2})$. Then, the resulting f is a history-based winning strategy of the cops team, and in particular, (`HS3`) is ensured by (`PS2`).

(2) We move to the other direction. Assume that the cops team has at least one history-based winning strategy.

First, let us consider a simple case that there is a history-based winning strategy g satisfying the following condition:

> for any two histories $h_1, h_2 \in \mathcal{H}^\mathcal{G}_\mathsf{C}$ whose last pairs are $(\sigma'_{2n+1},\Pi'_{2n+1})$ and $(\sigma'_{2m+1},\Pi'_{2m+1})$ respectively, when the information states $\Sigma^{2n+1}_\mathsf{C} \in \Pi'_{2n+1}$ and $\Sigma^{2m+1}_\mathsf{C} \in \Pi'_{2m+1}$ are the same, the cops need to take the same actions on the basis of the two histories. (†)

Then, based on such a g, we can construct a positional strategy f for the cops team satisfying the following: if there is a history $h = (\sigma'_1,\Pi'_1),\ldots,(\sigma'_{2n+1},\Pi'_{2n+1})$ such that $g(h)$ is obtained by adding $(\sigma'_{2n+2},\Pi'_{2n+2})$ to h, then

$$f(\sigma'_{2n+1},\Pi'_{2n+1}) := (\sigma'_{2n+2},\Pi'_{2n+2}).$$

The condition (†) is crucial, and it ensures that the resulting f is a function. Any such f satisfying (`PS1`) is a positional winning strategy for the cops team, and in particular, (`HS3`) ensures that it has the property (`PS2`).

Next, it suffices to prove that the cops team always has a history-based winning strategy satisfying (†). Fix an arbitrary history-based winning strategy g of the cops team. Suppose that the strategy does not satisfy (†), and we show how to modify it to satisfy the property.

Now there are different histories h_1, h_2 such that the information states $\Sigma^{2n+1}_\mathsf{C} \in \Pi_{2n+1}$ and $\Sigma^{2m+1}_\mathsf{C} \in \Pi_{2m+1}$ from their last pairs $(\sigma_{2n+1},\Pi_{2n+1})$ and $(\sigma_{2m+1},\Pi_{2m+1})$ are the same, but according to the strategy, the cops need to take different actions. For the g and the Σ^{2n+1}_C, we define $\mathtt{Action}^g_{\Sigma^{2n+1}_\mathsf{C}}$ as the set consisting of pairs (σ,Π) s.t. g adds (σ,Π) to a history, with an odd length, whose last pair has Σ^{2n+1}_C as the information state of the cops team. Intuitively, $\mathtt{Action}^g_{\Sigma^{2n+1}_\mathsf{C}}$ collects all possible movements of the cops based on Σ^{2n+1}_C under g, and it is a finite set. Moreover, there must be a situation σ from some pair in $\mathtt{Action}^g_{\Sigma^{2n+1}_\mathsf{C}}$

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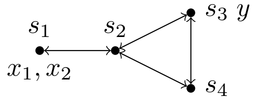
[5] Here two histories cannot be distinguished only when their lengths are the same. Games with this feature are called *von Neumann game* (Kuhn, 1997; Ghosh and Padmanabha, 2018). With respect to the game, we need to assume that *players can remember how their knowledge is updated in each step of the histories*.

such that if the cops act according to g, with the exception that they always move to $\sigma(x_1)$ and $\sigma(x_2)$ whenever the team has the information state $\Sigma^{2n+1}_{\mathsf{C}}$, then they can win no matter how the robber acts: otherwise, when $\Sigma^{2n+1}_{\mathsf{C}}$ is the information state of the cops team and it is their turn to act, any action of them according to g would leave a room for the robber to survive, in that there is a history $h \in \mathcal{H}^{\mathcal{G}}_{\mathsf{C}}$, considered to be possible by the cops team, such that its last pair has $\Sigma^{2n+1}_{\mathsf{C}}$ as the information state and it is the turn of the cops to act, but the cops team cannot win by performing that action. In such a way, we can always modify g to a winning strategy for the cops team that satisfies (†), as needed. ☐

4 Higher-order knowledge in games

As mentioned, the information states are about the first-order knowledge of the players, but contain no information about their higher-order knowledge. To take a closer look at the latter notion, we consider an example, which would indicate how higher-order knowledge can emerge:

Example 18. *In the picture below, both x_1, x_2 are at s_1, and y is at s_3.*



Let us consider the team knowledge of the cops. They know that y is at 3 or 4, but do not know which one is the case. For the higher-order knowledge, they would know that y knows where x_1 and x_2 are: from each of the positions s_3 and s_4 that are considered to be possible by cops, y can see the same region $\{s_2, s_3, s_4\}$, and he cannot see x_1 or x_2, which would make y know that both x_1 and x_2 are at s_1.

Precisely, the higher-order knowledge discussed in the example is *second-order knowledge*, i.e., the knowledge about the knowledge of positions. Arriving at the apparently reasonable conclusion about the higher-order knowledge in the example above also has its price:

Remark 19. *To ensure that we can obtain the conclusion that the cops team knows that y knows where x_1 and x_2 are, we need to enhance the assumption that* everyone knows *the game board structure (Remark 8) to a stronger form that* everyone knows that everyone knows *the structure. More generally, to fit with the setting with unlimited higher-order of knowledge, we need to work with the assumption that it is* common knowledge *that all players know the structure.*

In this section, we develop the game involving second-order knowledge - the definitions and results mentioned here can be generalized to arbitrarily higher orders of knowledge.

Definition 20. *A* cops and robber game with second-order knowledge *is a tuple $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi)$, where $(\mathbf{D}, \mathbf{R})$ is a game board, and $\Pi = \{\Theta_{\mathsf{C}}, \Theta_{\mathsf{R}}\}$ consists of the* information states $\Theta_{\mathsf{C}} = \{\mathbb{F}_{\mathsf{C}}, \mathbb{H}_{\mathsf{C}}\}$ *of the cops team and* $\Theta_{\mathsf{R}} = \{\mathbb{F}_{\mathsf{R}}, \mathbb{H}_{\mathsf{R}}\}$ *of the robber with the following properties:*

(`HG1`) $\mathbb{F}_{\mathsf{C}}, \mathbb{F}_{\mathsf{R}} \subseteq \mathbf{D}^{\mathsf{Var}}$ *are non-empty sets of situations, and satisfy the conditions* (`G1`), (`G2`) *and* (`G3`).
[$\mathbb{F}_{\mathsf{C}}$ and $\mathbb{F}_{\mathsf{R}}$ represent the first-order knowledge.]

(`HG2`) $\mathbb{H}_{\mathsf{C}} \subseteq \mathbb{F}_{\mathsf{C}} \times \mathcal{P}(\mathbf{D}^{\mathsf{Var}})$ *satisfies the following:*

(`HG2.1`) *For any $\sigma' \in \mathbb{F}_{\mathsf{C}}$, there is at least one $\Sigma \subseteq \mathbf{D}^{\mathsf{Var}}$ with $\langle \sigma', \Sigma \rangle \in \mathbb{H}_{\mathsf{C}}$.*

(`HG2.2`) *For any $\langle \sigma', \Sigma \rangle \in \mathbb{H}_{\mathsf{C}}$, Σ satisfies the requirement of* (`G1`) *for the 1-sight ability of the robber.*

[$\mathbb{H}_{\mathsf{C}}$ represents the knowledge of the cops team regarding the first-order knowledge of the robber y: (`HG2.1`) *means for each σ' considered to be possible by the team, there is at least a class of situations, representing y's first-order knowledge that is considered to be possible by the cops team; and by* (`HG2.2`)*, the cops know that y has the observational power.]*

(`HG3`) $\mathbb{H}_{\mathsf{R}} \subseteq \mathbb{F}_{\mathsf{R}} \times \mathcal{P}(\mathbf{D}^{\mathsf{Var}})$ *satisfies the following:*

(`HG3.1`) *For any $\sigma' \in \mathbb{F}_{\mathsf{R}}$, there is at least one $\Sigma \subseteq \mathbf{D}^{\mathsf{Var}}$ with $\langle \sigma', \Sigma \rangle \in \mathbb{H}_{\mathsf{R}}$.*

(`HG3.2`) *For any $\langle \sigma', \Sigma \rangle \in \mathbb{H}_{\mathsf{R}}$, Σ satisfies the requirement of* (`G2`) *for the 1-sight ability of the cops.*

[$\mathbb{H}_{\mathsf{R}}$ represents the knowledge of the robber regarding the first-order knowledge of the cops team.]

(`HG4`) *For any $\langle \sigma', \Sigma \rangle \in \mathbb{H}_{\mathsf{C}} \cup \mathbb{H}_{\mathsf{R}}$, $\sigma' \in \Sigma$. Moreover, when σ' is the actual situation (i.e., $\sigma' \in \mathbb{F}_{\mathsf{C}} \cap \mathbb{F}_{\mathsf{R}}$), it holds that $\langle \sigma', \mathbb{F}_{\mathsf{R}} \rangle \in \mathbb{H}_{\mathsf{C}}$ and $\langle \sigma', \mathbb{F}_{\mathsf{C}} \rangle \in \mathbb{H}_{\mathsf{R}}$.*
[Intuitively, the first part ensures that no players would ignore the 'actual situation', and the second part aims to ensure the veridicality of the second-order knowledge.]

Remark 21. *Compared with the first-order knowledge setting (Definition 3), we have some additional assumptions.*

- *As analyzed,* (`HG2.2`) *indicates that cops know that the robber has the 1-sight ability, and* (`HG3.2`) *indicates that the robber knows that each cop has the 1-sight ability.*
- *The second-order knowledge $\mathbb{H}_{\mathsf{R}}$ of the robber is given by pairs of the form $\langle \sigma, \Sigma \rangle$, meaning that the robber considers the following to be possible: when σ is the actual case, the cops team cannot distinguish between the situations in Σ. This indicates that the robber knows that the two cops share (first-order) knowledge.*

These assumptions will affect how the players can update their knowledge during the game play (see Definition 22).

Our purpose here is to propose a faithful framework for the game, and the method used to represent players' knowledge differs from standard epistemic logic (van Benthem, 2011; van Ditmarsch, van der Hoek, and Kooi, 2008; Fagin et al., 1995; Baltag, Moss, and Solecki, 1998). In the latter, what an agent cannot distinguish between are merely possible worlds, and such indistinguishability relations between those worlds are assumed to be common knowledge. By contrast, according to the definition above, a player may not know what others cannot distinguish: for instance, given a σ, there may exist distinct Σ and Σ' such that the cops team cannot distinguish between $\langle \sigma, \Sigma \rangle$ and $\langle \sigma, \Sigma' \rangle$, meaning that when σ is the actual situation, the team does not know what the robber cannot distinguish between. This can happen during the game play: for instance, starting from a fixed $\langle \sigma, \Sigma \rangle$, actions of players may lead to different $\langle \sigma', \Sigma_1 \rangle$ and $\langle \sigma', \Sigma_2 \rangle$ (cf. Example 23).

Definition 22. *Given a game with second-order knowledge $\mathcal{G}_1 = (\mathbf{D}, \mathbf{R}, \Pi_1 = \{\{\mathbb{F}^1_{\mathsf{C}}, \mathbb{H}^1_{\mathsf{C}}\}, \{\mathbb{F}^1_{\mathsf{R}}, \mathbb{H}^1_{\mathsf{R}}\}\}, \sigma_1)$, an* action *(or* movement*) of $\star \in \{\mathsf{C}, \mathsf{R}\}$ from $\mathcal{G}_1$ updates it into a game $\mathcal{G}_2 = (\mathbf{D}, \mathbf{R}, \Pi_2 = \{\{\mathbb{F}^2_{\mathsf{C}}, \mathbb{H}^2_{\mathsf{C}}\}, \{\mathbb{F}^2_{\mathsf{R}}, \mathbb{H}^2_{\mathsf{R}}\}\}, \sigma_2)$, where*

(`HAct1`) *$\sigma_2 \in \mathsf{M}^\star(\sigma_1)$, and $\mathbb{F}^2_{\mathsf{R}}$ and $\mathbb{F}^2_{\mathsf{C}}$ are given by* (`Act2`) *and* (`Act3`) *in Definition 6, respectively.*

(`HAct2`) *$\mathbb{H}^2_{\mathsf{C}}$ consists of those $\langle \sigma'_2, \Sigma'_2 \rangle \subseteq \mathbb{F}^2_{\mathsf{C}} \times \mathcal{P}(\mathbf{D}^{\mathsf{Var}})$ such that there is a $\langle \sigma'_1, \Sigma'_1 \rangle \in \mathbb{H}^1_{\mathsf{C}}$ as follows:*

(`HAct2.1`) *$\sigma'_2 \in \mathbb{F}^2_{\mathsf{C}} \cap \mathsf{M}^\star(\sigma'_1)$.*

(`HAct2.2`) *Σ'_2 consists of the situations $\sigma \in \mathsf{M}^\star(\Sigma'_1)$ such that for any $v \in \mathsf{Var}$,*

- *If $\sigma'_2(v) \in \mathbb{S}(\sigma'_2(y))$, then $\sigma(v) = \sigma'_2(v)$.*
- *If $\sigma'_2(v) \notin \mathbb{S}(\sigma'_2(y))$, then $\sigma(v) \notin \mathbb{S}(\sigma(y))$.*

[It concerns how second-order knowledge of the cops team is updated. Those new $\langle \sigma'_2, \Sigma'_2 \rangle$ are obtained from the old ones $\langle \sigma'_1, \Sigma'_1 \rangle$. By the first sub-clause, σ'_2 should be obtainable from a previous σ'_1 of $\langle \sigma'_1, \Sigma'_1 \rangle$ via $\star$'s movement along $\mathbf{R}$, and σ'_2 itself is considered to be possible by the cops team after the action. By the second sub-clause, the cops team knows that the robber updates his first-order knowledge based on his 1-sight ability.]

(`HAct3`) *$\mathbb{H}^2_{\mathsf{R}}$ consists of those $\langle \sigma'_2, \Sigma'_2 \rangle \subseteq \mathbb{F}^2_{\mathsf{R}} \times \mathcal{P}(\mathbf{D}^{\mathsf{Var}})$ such that there is a pair $\langle \sigma'_1, \Sigma'_1 \rangle \in \mathbb{H}^1_{\mathsf{R}}$ as follows:*

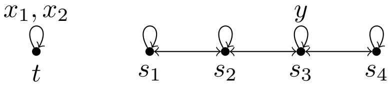
(`HAct3.1`) *$\sigma'_2 \in \mathbb{F}^2_{\mathsf{R}} \cap \mathsf{M}^\star(\sigma'_1)$.*

(`HAct3.2`) *Σ'_2 consists of those $\sigma \in \mathsf{M}^\star(\Sigma'_1)$ such that*

- *For any $x \in \mathsf{C}$, $\sigma(x) = \sigma'_2(x)$.*
- *If $\sigma'_2(y) \in \mathbb{S}(\sigma'_2(x_1)) \cup \mathbb{S}(\sigma'_2(x_2))$, then $\sigma = \sigma'_2$.*
- *If $\sigma'_2(y) \notin \mathbb{S}(\sigma'_2(x_1)) \cup \mathbb{S}(\sigma'_2(x_2))$, then it holds that $\sigma(y) \notin \mathbb{S}(\sigma(x_1)) \cup \mathbb{S}(\sigma(x_2))$.*

[It concerns how y's second-order knowledge is updated. The second and the third sub-clauses indicate that y knows that the cops share information with each other.]

Example 23. *We consider an example depicted below.*



From the perspective of the cops team, any of s_1-s_4 could be the position of y. Also, the team would think the following is the case, involving the first-order knowledge of y: if y is at s_2, then y would know that cops are at t or s_4. So, the higher-order information state $\mathbb{H}^1_{\mathsf{C}}$ at this stage contains

$$\langle (t, t, s_2), \Sigma_1 = \{(t, t, s_2), (t, s_4, s_2), (s_4, t, s_2), (s_4, s_4, s_2)\} \rangle.$$

Assume that y moved to s_2 in the first round, and moved to s_3 in the second round. Let us consider the information states of the cops team at the end of the second round. Regarding their first-order knowledge, the team cannot rule out any possibilities, and with respect to their higher-order knowledge, for instance, their higher-order information state $\mathbb{H}^5_{\mathsf{C}}$ at this stage contains not only $\langle (t, t, s_2), \Sigma_1 \} \rangle$, but also $\langle (t, t, s_2), \{(t, t, s_2)\} \rangle$: from the perspective of cops, it is possible that the initial position of the robber is s_2, and based on this, both the cases below are possible by the end of the second round:

- *The robber stayed at s_2 in the first two rounds, and did not gain any new information (so $\langle (t, t, s_2), \Sigma_1 \} \rangle \in \mathbb{H}^5_{\mathsf{C}}$).*
- *The robber moved to s_3 in the first round and returned to s_2 in the second round. In such a case, y knew that both the cops are at t (so, $\langle (t, t, s_2), \{(t, t, s_2)\} \rangle \in \mathbb{H}^5_{\mathsf{C}}$).*

Remark 24. *For the setting with second-order knowledge, we should enhance the assumptions stated in Remark 8:*

- *After an action of cops, cops update not only their first-order knowledge, but also their second-order knowledge about the first-order knowledge of the robber y. This indicates that cops know that y knows it was the turn of cops to act. Similarly, when it is the turn of y to act, y also knows that cops know the fact. Thus, the cops and the robber know that each other knows whose turn it is to act.*
- *The way to update second-order knowledge demonstrates that the cops and the robber know that each other knows the game board structure.*
- *After an action, the cops and the robber update both their first-order and second-order knowledge with the previous ones, which indicates that they know that each other does not forget the* first-order knowledge *before the action.*

Proposition 25. *Let $\mathcal{G}_1 = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ be a game with second-order knowledge, $\star \in \{\mathsf{C}, \mathsf{R}\}$ and $\sigma_2 \in \mathsf{M}^\star(\sigma_1)$. Then, $\mathcal{G}_2 = (\mathbf{D}, \mathbf{R}, \Pi_2, \sigma_2)$ is well defined, where Π_2 is given by* (`HAct1`)-(`HAct3`). *Moreover, $\{\sigma_2\} = \mathbb{F}^2_{\mathsf{C}} \cap \mathbb{F}^2_{\mathsf{R}}$.*

Proof. The argument for $\{\sigma_2\} = \mathbb{F}^2_{\mathsf{C}} \cap \mathbb{F}^2_{\mathsf{R}}$ is the same as before (Proposition 9). In what follows, we prove that the resulting $\mathcal{G}_2$ is well defined. To achieve this, we will show that (`HG1`)-(`HG4`) from Definition 20 are satisfied.

(1) First, it follows from (`HAct1`) that (`HG1`) is satisfied. The detailed argument for this is similar to that in the proof for Proposition 9, and we skip the details.

(2) Next, we prove that (`HG2`) and (`HG3`) hold. They are analogous, and we only consider the former. By (`HAct2.1`), $\langle \sigma', \Sigma \rangle \in \mathbb{H}^2_{\mathsf{C}}$ implies $\sigma' \in \mathbb{F}^2_{\mathsf{C}}$. So, $\mathbb{H}^2_{\mathsf{C}} \subseteq \mathbb{F}^2_{\mathsf{C}} \times \mathcal{P}(\mathbf{D}^{\mathsf{Var}})$.

Now we show that (`HG2.1`) and (`HG2.2`) hold. Let us start with the former. Assume that $\sigma'_2 \in \mathbb{F}^2_{\mathsf{C}}$. Then, $\sigma'_2 \in \mathsf{M}^\star(\mathbb{F}^1_{\mathsf{C}})$. So, there is a situation $\sigma'_1 \in \mathbb{F}^1_{\mathsf{C}}$ such that $\mathsf{M}^\star \sigma'_1 \sigma'_2$. Fix such a situation σ'_1. Due to the well-definiteness of $\mathcal{G}_1$, there is a set Σ'_1 such that $\langle \sigma'_1, \Sigma'_1 \rangle \in \mathbb{H}^1_{\mathsf{C}}$. We write Σ'_2 for the subset of $\mathsf{M}^\star(\Sigma'_1)$ that consists of the situations satisfying the two conditions imposed in (`HAct2.2`). Notice that Σ'_2 is not empty: by (`HG4`), it holds that $\sigma'_1 \in \Sigma'_1$, and so $\sigma'_2 \in \Sigma'_2$. Now we have $\langle \sigma'_2, \Sigma'_2 \rangle \in \mathbb{H}^2_{\mathsf{C}}$, which indicates that (`HG2.1`) is satisfied. Moreover, to see that (`HG2.2`) is also satisfied, note that the two conditions imposed in (`HAct2.2`) ensures that the robber has the 1-sight ability, i.e., Σ'_2 satisfies (`G1`).

(3) Finally, we show that (`HG4`) is satisfied. We know from the reasoning above that for any $\langle \sigma'_2, \Sigma'_2 \rangle \in \mathbb{H}_{\mathsf{C}} \cup \mathbb{H}_{\mathsf{R}}$, $\sigma'_2 \in \Sigma'_2$. Moreover, recall that σ_2 is the new actual situation, and we prove that $\langle \sigma_2, \mathbb{F}^2_{\mathsf{R}} \rangle \in \mathbb{H}^2_{\mathsf{C}}$ and $\langle \sigma_2, \mathbb{F}^2_{\mathsf{C}} \rangle \in \mathbb{H}^2_{\mathsf{R}}$. They are analogous, and we just consider the latter. Note that $\langle \sigma_1, \mathbb{F}^1_{\mathsf{C}} \rangle \in \mathbb{H}^1_{\mathsf{R}}$. By (`HAct3`), we have $\langle \sigma_2, \mathbb{F}^2_{\mathsf{C}} \rangle \in \mathbb{H}^2_{\mathsf{R}}$. ☐

5 Does second-order knowledge matter?

A 'run' for the game with second-order knowledge can be defined in the same way as that for the first-order knowledge setting (Definition 10), except that the information states Π involved in a sequence of pairs (σ, Π) now also contain the information about second-order knowledge. The mechanism for determining whether the cops team knows the exact position of the robber is given by their first-order knowledge $\mathbb{F}_{\mathsf{C}}$. As before, given a run, the cops team wins if at some stage $\mathbb{F}_{\mathsf{C}}$ is a singleton set; otherwise the robber wins.

As the setting with only first-order knowledge (Section 3), players in a game $\mathcal{G}$ with the second-order knowledge may not know which game they will play, either. Analogous to Definition 11, we can define $\mathbb{G}^{\mathcal{G}}_{\mathsf{C}}$ and $\mathbb{G}^{\mathcal{G}}_{\mathsf{R}}$ that are considered to be possible by the cops team and the robber, respectively.

Moreover, we can extend the notions of history-based strategies, positional strategies and the corresponding winning strategies to the higher-order knowledge setting. To save space, we only present the positional ones.

Definition 26. *Let $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi_1 = \{\Theta^1_{\mathsf{C}}, \Theta^1_{\mathsf{R}}\}, \sigma_1)$ be a game with second-order knowledge. Let $\mathbb{P}$ be the set consisting of the pairs (σ, Π) such that Π satisfies* `(HG1)`-`(HG4)` *and contains $\mathbb{F}_{\mathsf{C}}, \mathbb{F}_{\mathsf{R}}$ as the first-order knowledge of the players.*

A positional strategy for the cops team *is a function $f : \mathbb{P} \to \mathbb{P}$ such that*

- `(HPS1)` *$f(\sigma, \Pi) = (\sigma', \Pi')$ only if $\mathbb{F}_{\mathsf{C}}$ from Π is not a singleton set, $\sigma' \in \mathsf{M}^{\mathsf{C}}(\sigma)$, and Π' is the class of information states in the game $(\mathbf{D}, \mathbf{R}, \Pi', \sigma')$ obtained by updating $(\mathbf{D}, \mathbf{R}, \Pi, \sigma)$ with the movement of cops from $\sigma(x_1)$ and $\sigma(x_2)$ to, respectively, $\sigma'(x_1)$ and $\sigma'(x_2)$.*
- `(HPS2)` *For any pairs $(\sigma_a, \Pi_a), (\sigma_b, \Pi_b), (\sigma_c, \Pi_c), (\sigma_d, \Pi_d)$ from $\mathbb{P}$, when $\Theta^a_{\mathsf{C}} \in \Pi_a$ and $\Theta^c_{\mathsf{C}} \in \Pi_c$ are the same, $f(\sigma_a, \Pi_a) = (\sigma_b, \Pi_b)$ and $f(\sigma_c, \Pi_c) = (\sigma_d, \Pi_d)$, it holds that $\sigma_b(x) = \sigma_d(x)$ for any $x \in \mathsf{C}$.*

Similarly, a positional strategy for the robber *is also a function $g : \mathbb{P} \to \mathbb{P}$ such that*

- `(HPS3)` *$g(\sigma, \Pi) = (\sigma', \Pi')$ only if $\mathbb{F}_{\mathsf{C}}$ from Π is not a singleton set, $\sigma' \in \mathsf{M}^{\mathsf{R}}(\sigma)$, and Π' is the class of information states in the game $(\mathbf{D}, \mathbf{R}, \Pi', \sigma')$ obtained by updating $(\mathbf{D}, \mathbf{R}, \Pi, \sigma)$ with the movement of y from $\sigma(y)$ to $\sigma'(y)$.*
- `(HPS4)` *For any pairs $(\sigma_a, \Pi_a), (\sigma_b, \Pi_b), (\sigma_c, \Pi_c), (\sigma_d, \Pi_d)$ from $\mathbb{P}$, when $\Theta^a_{\mathsf{R}} \in \Pi_a$ and $\Theta^c_{\mathsf{R}} \in \Pi_c$ are the same, $g(\sigma_a, \Pi_a) = (\sigma_b, \Pi_b)$ and $g(\sigma_c, \Pi_c) = (\sigma_d, \Pi_d)$, it holds that $\sigma_b(y) = \sigma_d(y)$.*

We can generalize the previous notion of *initial game* for the first-order knowledge setting to the new framework. With this, we now define the notion of positional strategy:

Definition 27. *Let $\mathcal{G} = (\mathbf{D}, \mathbf{R}, \Pi_1, \sigma_1)$ be an initial game with second-order knowledge. A* positional winning strategy for the cops team *is a strategy such that when they follow it, they can win all runs of the games in $\mathbb{G}^{\mathcal{G}}_{\mathsf{C}}$. A* positional winning strategy for the robber *is a strategy such that when he follows it, he can win all runs of the games in $\mathbb{G}^{\mathcal{G}}_{\mathsf{R}}$.*

Histories do not matter Again, for the setting with higher-order knowledge, we can show the following:

Theorem 28. *Given an initial game with higher-order knowledge, it holds that*

- (i) *The cops team has a history-based winning strategy iff the team has a positional winning strategy.*
- (ii) *The robber has a history-based winning strategy iff he has a positional winning strategy.*

Higher-order knowledge does not matter We are finally ready to prove our main hypothesis: the players having second-order knowledge in the game do not affect the existence of winning strategies. In what follows, given a game $\mathcal{G}^{\mathsf{s}} = (\mathbf{D}, \mathbf{R}, \Pi^{\mathsf{s}})$ with second-order knowledge and a game $\mathcal{G}^{\mathsf{f}} = (\mathbf{D}, \mathbf{R}, \Pi^{\mathsf{f}})$ with first-order knowledge, when the first-order knowledge $\mathbb{F}_{\mathsf{C}}$ and $\mathbb{F}_{\mathsf{R}}$ from Π^{s} are the same as those from Π^{f}, we say that Π^{s} *is an extension of* Π^{f}. Also, we say that $\mathcal{G}^{\mathsf{s}}$ *is an extension of* $\mathcal{G}^{\mathsf{f}}$ if Π^{s} is an extension of Π^{f}.

Theorem 29. *Let $\mathcal{G}^{\mathsf{s}}$ be an initial game with second-order knowledge and $\mathcal{G}^{\mathsf{f}}$ an initial game with first-order knowledge such that $\mathcal{G}^{\mathsf{s}}$ is an extension of $\mathcal{G}^{\mathsf{f}}$.*

- (i) *The cops team has a positional winning strategy in $\mathcal{G}^{\mathsf{s}}$ iff the team has a positional winning strategy in $\mathcal{G}^{\mathsf{f}}$.*
- (ii) *The robber has a positional winning strategy in $\mathcal{G}^{\mathsf{s}}$ iff he has a positional winning strategy in $\mathcal{G}^{\mathsf{f}}$.*

Proof. The two parts are similar, and we just consider the first part. Since $\mathcal{G}^{\mathsf{s}} = (\mathbf{D}, \mathbf{R}, \Pi^{\mathsf{s}}_1)$ is an extension of $\mathcal{G}^{\mathsf{f}} = (\mathbf{D}, \mathbf{R}, \Pi^{\mathsf{f}}_1)$, they have the same actual situation. We write σ_1 for it. To avoid confusion, we will use '`f`' (first) and '`s`' (second) as superscripts to indicate the kind of game under consideration. For instance, we write Π^{s} to mean the set of information states of a game with second-order knowledge.

(1) We first assume that the cops team has a positional winning strategy $f : \mathbb{P}^{\mathsf{f}} \to \mathbb{P}^{\mathsf{f}}$ in $\mathcal{G}^{\mathsf{f}}$, where $\mathbb{P}^{\mathsf{f}}$ is given by the $\mathbb{P}$ in Definition 12. Based on f, we define a positional strategy $g : \mathbb{P}^{\mathsf{s}} \to \mathbb{P}^{\mathsf{s}}$ for the cops team in $\mathcal{G}^{\mathsf{s}}$ as follows:

$$g(\sigma_a, \Pi^{\mathsf{s}}_a) = (\sigma_b, \Pi^{\mathsf{s}}_b) \text{ iff } f(\sigma_a, \Pi^{\mathsf{f}}_a) = (\sigma_b, \Pi^{\mathsf{f}}_b),$$

where $\mathbb{P}^{\mathsf{s}}$ is given by the $\mathbb{P}$ in Definition 26, Π^{s}_a is an extension of Π^{f}_a, and Π^{s}_b is given by the movement of cops from $\sigma_a(x_1)$ and $\sigma_a(x_2)$ to $\sigma_b(x_1)$ and $\sigma_b(x_2)$, respectively. Clearly, Π^{s}_b is an extension of Π^{f}_b.

The resulting g is well defined, and one can check that g is a winning strategy for the cops team in $\mathcal{G}^{\mathsf{s}}$.

(2) We now consider the converse. Assume that the cops team has at least one positional winning strategy in $\mathcal{G}^{\mathsf{s}}$.

We starting by considering the case that there is a strategy g that satisfies the following property:

> for any $(\sigma_a, \Pi^{\mathsf{s}}_a)$ and $(\sigma_b, \Pi^{\mathsf{s}}_b)$ with the same components for the first-order team knowledge of the cops (i.e., $\mathbb{F}^a_{\mathsf{C}} = \mathbb{F}^b_{\mathsf{C}}$), it holds that $\sigma_c(x) = \sigma_d(x)$ for any cop $x \in \mathsf{C}$, where $g(\sigma_a, \Pi^{\mathsf{s}}_a) = (\sigma_c, \Pi^{\mathsf{s}}_c)$ and $g(\sigma_b, \Pi^{\mathsf{s}}_b) = (\sigma_d, \Pi^{\mathsf{s}}_d)$. (‡)

Based on such a strategy g, we can define a strategy f for the cops team in $\mathcal{G}^{\mathsf{f}}$ as follows: $f(\sigma_a, \Pi^{\mathsf{f}}_a) = (\sigma_b, \Pi^{\mathsf{f}}_b)$ iff there are the extensions $(\sigma_a, \Pi^{\mathsf{s}}_a)$ and $(\sigma_b, \Pi^{\mathsf{s}}_b)$ of $(\sigma_a, \Pi^{\mathsf{f}}_a)$ and

$(\sigma_b, \Pi^{\mathsf{f}}_b)$, respectively, such that $g(\sigma_a, \Pi^{\mathsf{s}}_a) = (\sigma_b, \Pi^{\mathsf{s}}_b)$. The condition (‡) ensures that f is a function, which together with (`HPS2`) also guarantees that f satisfies (`PS2`). One can check that f is a winning strategy for the cops team.

Next, we show that the cops team always has a positional winning strategy enjoying the property (‡). Fix an arbitrary winning strategy g of the team that does not satisfy (‡), and we show how to modify it to satisfy the property.

Now, there are different $(\sigma_a, \Pi^{\mathsf{s}}_a)$ and $(\sigma_b, \Pi^{\mathsf{s}}_b)$ s.t. the components $\mathbb{F}^a_{\mathsf{C}}$ and $\mathbb{F}^b_{\mathsf{C}}$ for the first-order team knowledge of the cops are the same, but cops need to take different actions according to g. For the g and the $\mathbb{F}^a_{\mathsf{C}}$, we define $\mathsf{Action}^g_{\mathbb{F}^a_{\mathsf{C}}}$ as the set consisting of pairs $(\sigma, \Pi^{\mathsf{s}})$ s.t. g maps a pair $(\sigma_i, \Pi^{\mathsf{s}}_i)$ to $(\sigma, \Pi^{\mathsf{s}})$, where $\mathbb{F}^a_{\mathsf{C}}$ is also the first-order team knowledge in Π^{s}_i. Notice that $\mathsf{Action}^g_{\mathbb{F}^a_{\mathsf{C}}}$ is finite. Also, there must be a σ from some pair $(\sigma, \Pi^{\mathsf{s}})$ in $\mathsf{Action}^g_{\mathbb{F}^a_{\mathsf{C}}}$ s.t. if cops act according to g, with the exception that they always move to $\sigma(x_1)$ and $\sigma(x_2)$ whenever they have the first-order knowledge $\mathbb{F}^a_{\mathsf{C}}$, then they can win no matter how the robber acts: otherwise, when it is the turn of the cops to act and their information state for the first-order knowledge is $\mathbb{F}^a_{\mathsf{C}}$, any action of the cops according to g would leave a room for the robber to survive, in that there is a run that it is considered to be possible by the cops team and contains a stage such that the cops need to act and have $\mathbb{F}^a_{\mathsf{C}}$ as their first-order knowledge, but they cannot win in the run by performing that action.

In such a way, we can always modify an arbitrary winning strategy of the cops to a winning strategy satisfying (‡). □

6 Conclusion

In the paper, we explore an imperfect information game of cops and robber. We identify the natural assumptions regarding the players' knowledge and ability that are used in such games to make these notions explicit. Moreover, we prove that regarding the existence of winning strategies, only factual knowledge matters, which refers to the current information states for first-order knowledge: the historical information and higher-order knowledge do not increase the strategic ability in any way. Such considerations also make sense for the perfect information game setting, which usually assumes that players know each other's positions but does not consider higher-order knowledge.

A natural next step is to study whether the various game structures presented in the article have some influence on the complexity bound of finding winning strategies. Also, it is important to have a formal system to reason about such game dynamics along the lines of (Li, Ghosh, and Liu, 2025; Li et al., 2023). Moreover, as stated, our approach to the game aims to capture the knowledge of players in a faithful way, but it is also meaningful to develop the game based on the standard epistemic approach and make a comparison between the two directions (van Benthem, 2026). Finally, it is worth exploring further variants of the game. For instance, one can assume that the robber always knows the positions of the cops, whereas the cops may only have partial information. Alternatively, we may require that the cops win if at least one cop is at the same position as the robber.

References

Baltag, A.; Moss, L.; and Solecki, S. 1998. The logic of public announcements, common knowledge, and private suspicions. In Gilboa, I., ed., *Proceedings of TARK '1998*, 43–56. San Francisco: Morgan Kaufmann.

Bonato, A., and Nowakowski, R. J. 2011. *The Game of Cops and Robbers on Graphs*, volume 61 of *The Student Mathematical Library*. Providence: AMS.

Clarke, N. E.; Cox, D.; Duffy, C.; Dyer, D.; Fitzpatrick, S. L.; and Messinger, M. E. 2020. Limited visibility cops and robber. *Discrete Applied Mathematics* 282:53–64.

Fagin, R.; Halpern, J.; Moses, Y.; and Vardi, M. 1995. *Reasoning about Knowledge*. Cambridge: The MIT Press.

Ghosh, S., and Padmanabha, A. 2018. Revisiting games in dynamic-epistemic logic. In *Proceedings of LOFT 2018*.

Halpern, J. Y. 2007. Computer science and game theory: A brief survey. arXiv preprint cs/0703148.

Kuhn, H. 1997. *Classics in Game Theory*. Princeton University Press.

Li, D.; Ghosh, S.; Liu, F.; and Tu, Y. 2023. A simple logic of the hide and seek game. *Studia Logica* 111:821–853.

Li, D.; Ghosh, S.; and Liu, F. 2025. Reasoning under uncertainty in the game of Cops and Robbers. *Synthese* 206:101.

Nowakowski, R., and Winkler, P. 1983. Vertex-to-vertex pursuit in a graph. *Discrete Math* 13:235–239.

Quillot, A. 1978. *Jeux et pointes fixes sur les graphes*. Ph.D. Dissertation, Université de Paris VI.

Tang, A. 2004. Cops and robber with bounded visibility. Master's thesis, Dalhousie University.

Tošić, R. 1985. Vertex-to-vertex search in a graph. In *Proceedings of the Sixth Yugoslav Seminar on Graph Theory*, 233–237.

van Benthem, J. 2011. *Logical Dynamics of Information and Interaction*. Cambridge: Cambridge University Press.

van Benthem, J. 2026. Dynamic-epistemic logic for games. In van Benthem, J., and Liu, F., eds., *Graph Games and Logic Design: Recent Developments and Future Directions*. Cham: Springer. 339–357.

van Ditmarsch, H.; van der Hoek, W.; and Kooi, B. 2008. *Dynamic Epistemic Logic*. Dordrecht: Springer.